The Birch and Swinnerton-Dyer conjecture

Christian Wuthrich

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Our main question

How can we determine the set of solutions $E(K)$ with coordinates in K ?

Question

Are there infinitely many rational solutions to E over $\mathbb{Q}$?

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has only three solutions $(-1, 0)$, $(1, -2)$, and $(1, 2)$.

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The following x-coordinates are

$$-\frac{287}{1296}, \quad \frac{43992}{82369}, \quad \frac{26862913}{1493284}, \quad \frac{139455877527}{1824793048}, \quad -\frac{3596697936}{8760772801},$$
$$\frac{7549090222465}{8662944250944}, \quad \frac{51865013741670864}{6504992707996225}, \quad -\frac{173161424238594532415}{3105156367744812388 84}, \quad \ldots$$

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Addition on elliptic curves

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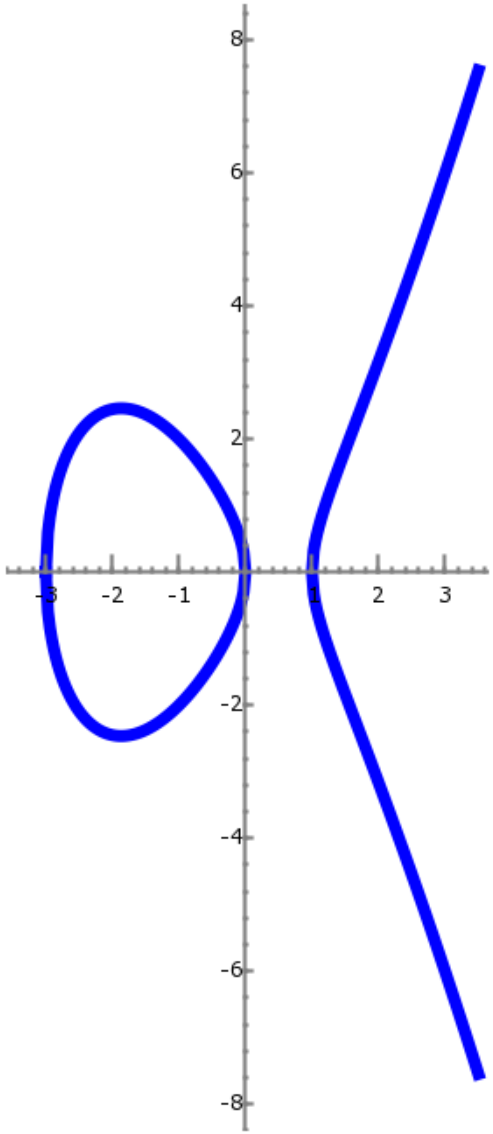

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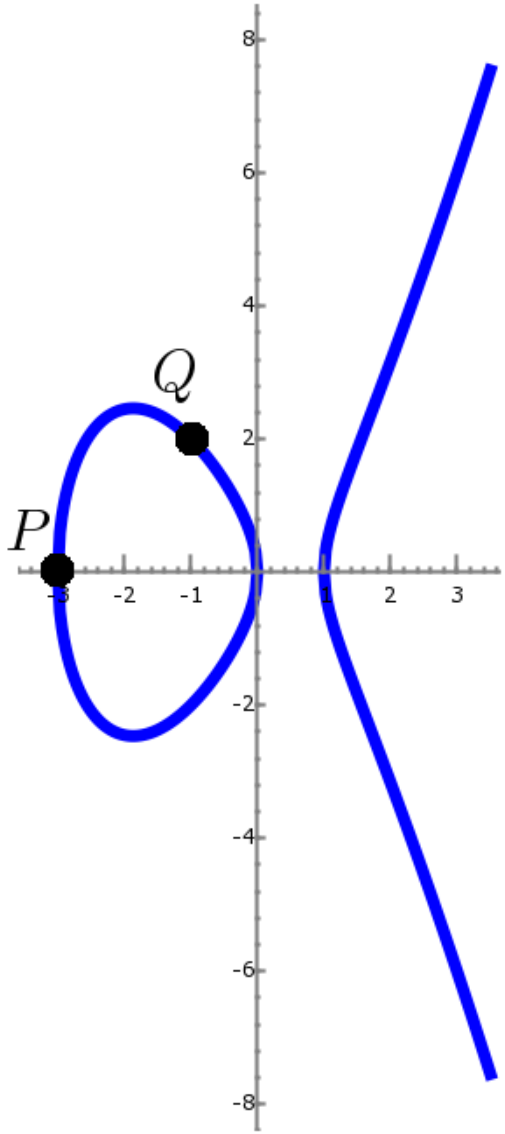
Any two points P and Q on E



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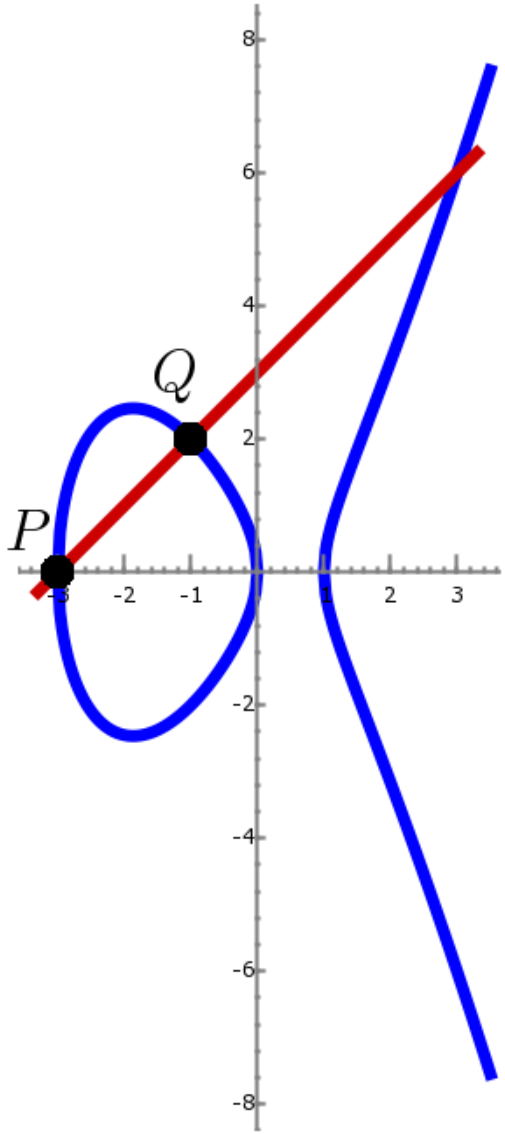

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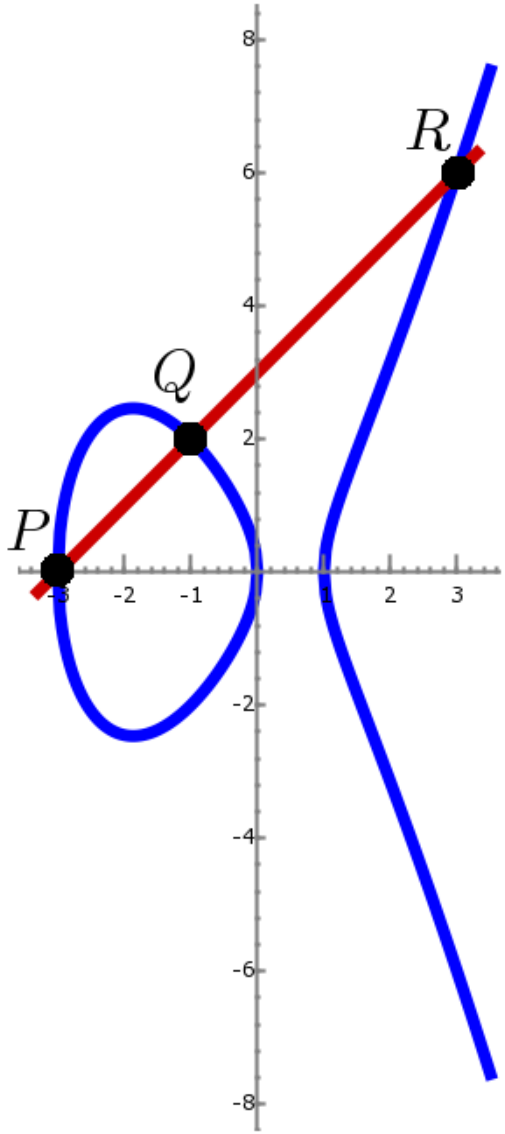

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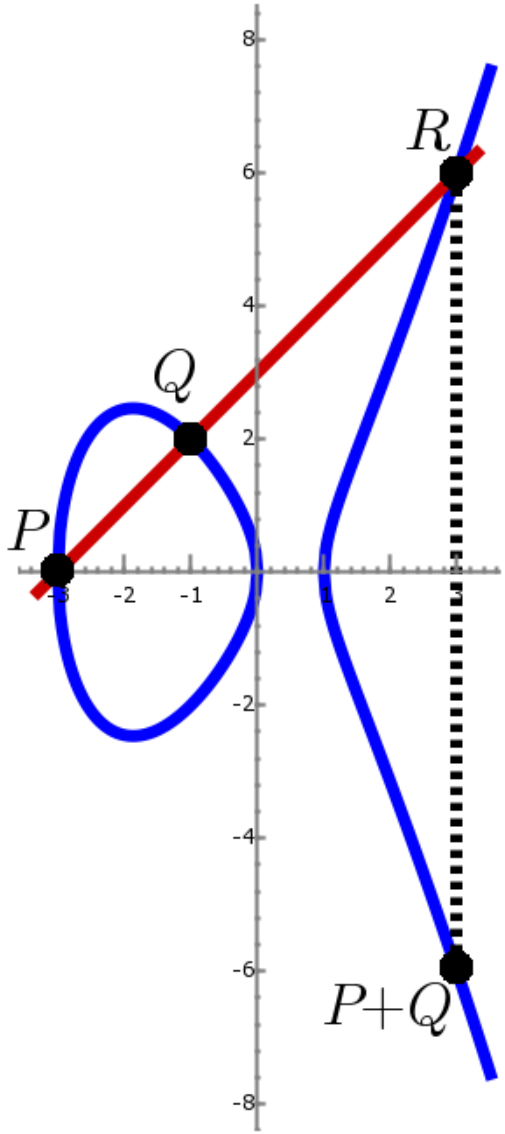

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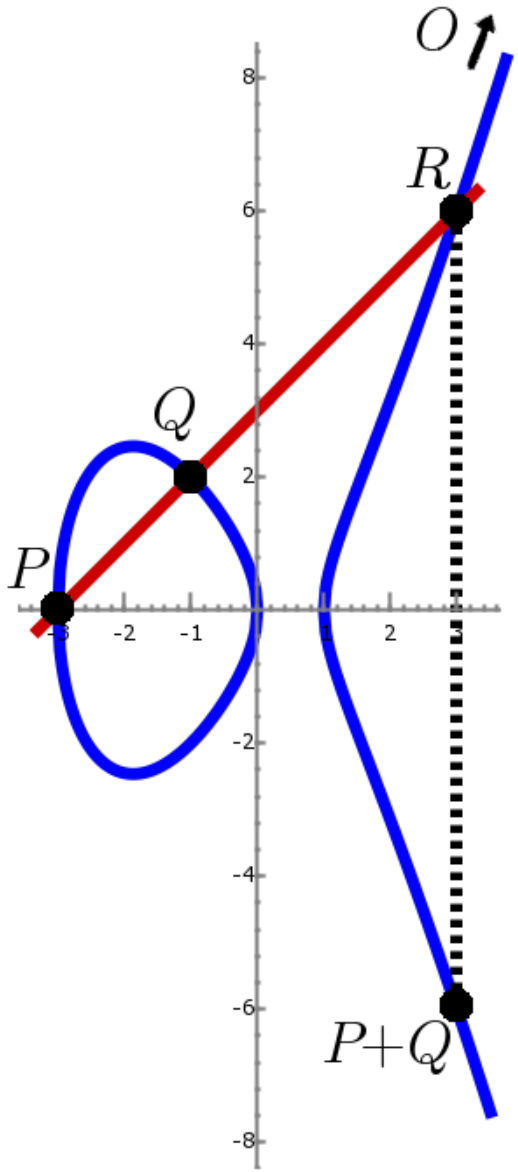

Elliptic curves over finite fields

Hasse-Weil bound

An elliptic curve E over $\mathbb{F}_p$ satisfies

$$N_p = \#E(\mathbb{F}_p) = p + 1 - a_p$$

with $|a_p| < 2\sqrt{p}$.

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Curve sepc160k1

$$E: \ y^2 = x^3 + 7 \qquad \text{with}$$

$$p = 1461501637330902918203684832716283019651637554291$$

$$N_p = 1461501637330902918203686915170869725397159163571$$

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- E_2 has rank 0 and $E_2(\mathbb{Q}) = {}^{\mathbb{Z}}\!/_{4\mathbb{Z}}\,(1,2)$, while
- E_1 has rank 1 and $E_1(\mathbb{Q}) = \mathbb{Z}\,(0,1)$.

Bryan Birch and Sir Peter Swinnerton-Dyer

Let E be an elliptic curve over $\mathbb{Q}$ with $A, B \in \mathbb{Z}$.

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Conjecture

$f(X)$ stays bounded if and only if there are only finitely many solutions in $\mathbb{Q}$.

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$f(X)$ grows like $r \cdot \log(\log(X))$, where r is the rank of $E(\mathbb{Q})$.

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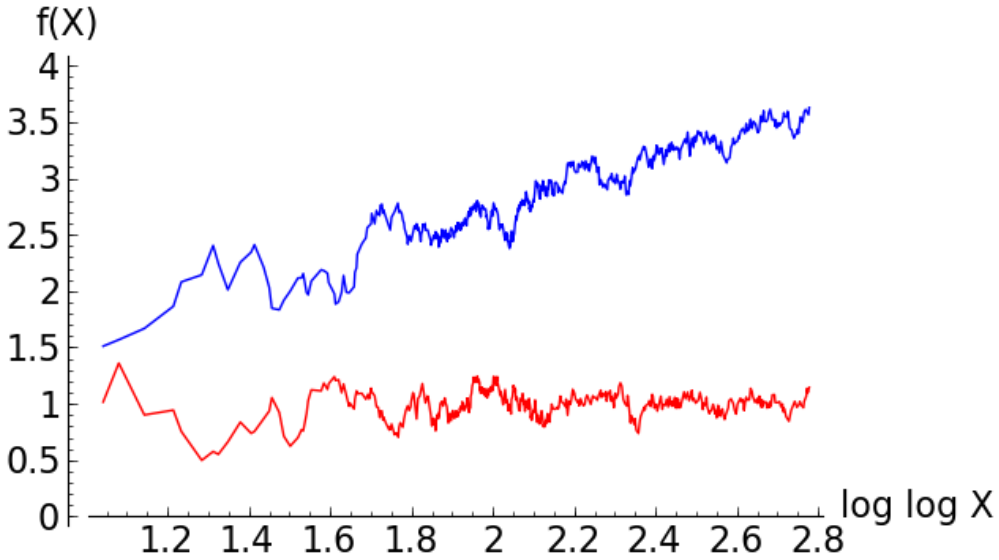

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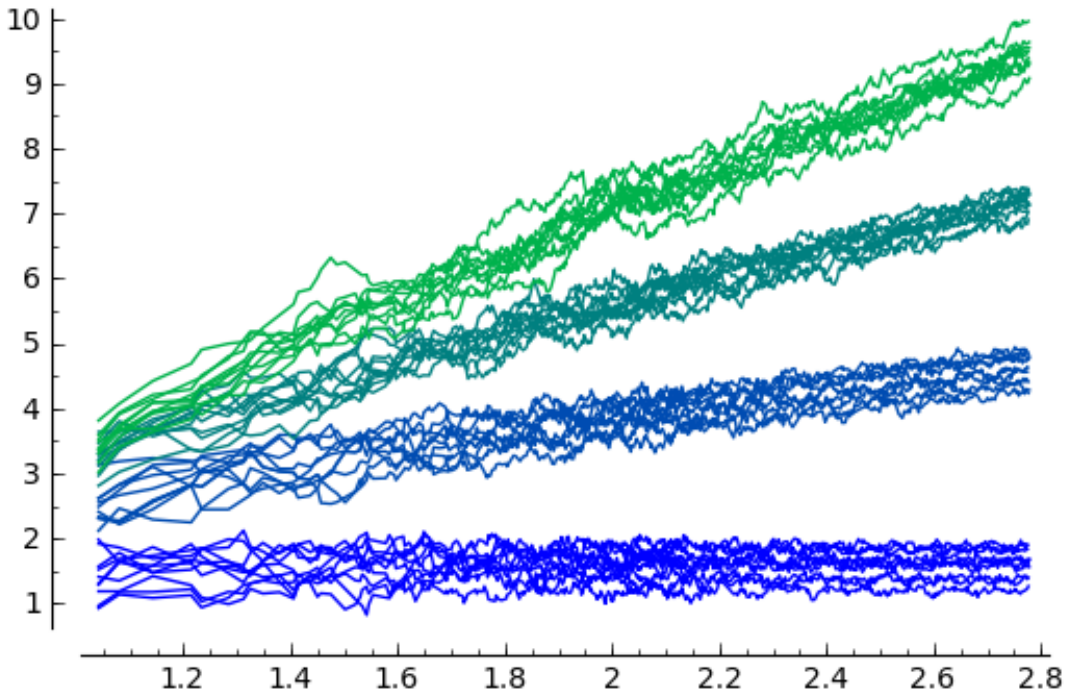

The L-series

Define

$$L(E,s) = \prod_{p \text{ good}} \frac{1}{1 - a_p \cdot p^{-s} + p \cdot p^{-2s}} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

for $\mathrm{Re}(s) > \frac{3}{2}$.

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$$\text{“ } L(E,1) = \prod_{p} \frac{p}{N_p} = \exp\big(-f(\infty)\big) \text{ ”.}$$

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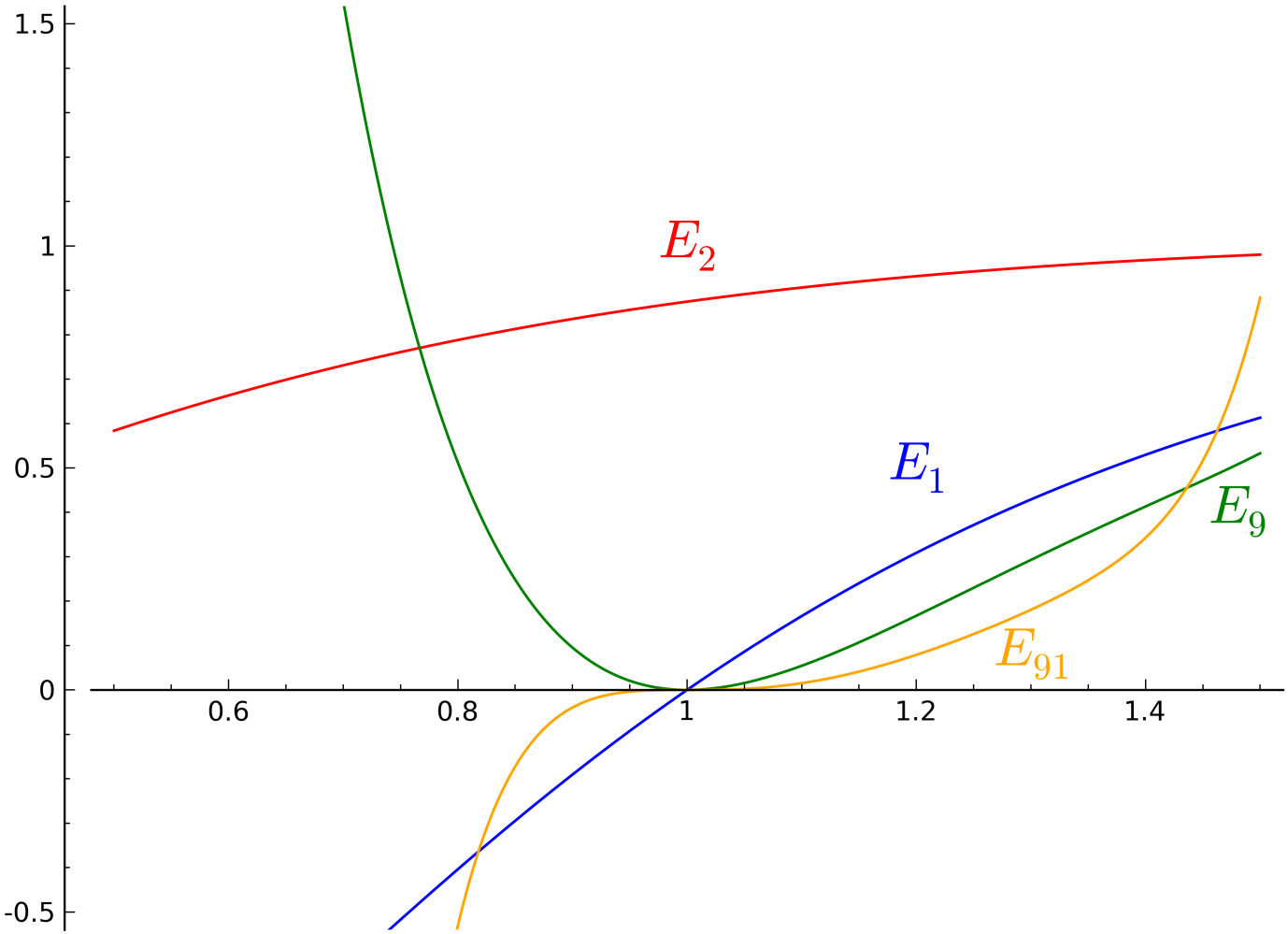
Weak Birch and Swinnerton-Dyer conjecture 1000000$

The function $L(E,s)$ has a zero of order r, the rank of $E(\mathbb{Q})$, at $s = 1$.


1.5
1
0.5
0
-0.5
0.6
0.8
1
1.2
1.4
E_2
E_1
E_9
E_{91}

Results

Taylor-Wiles et al.

If $E/\mathbb{Q}$, then $L(E,s)$ has an analytic continuation to $\mathbb{C}$.
In fact, $L(E,s) = L(f,s)$ for a modular form f.

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If $r_{\text{an}} > 1$, ???

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which converges for $\mathrm{Re}(s) > \frac{3}{2}$.

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Nekovář, T&V Dokchitser

If . . . , then $\mathrm{ord}_{s=1} L(E/K,s) \equiv \mathrm{rank}\, E(K) \pmod 2$

The conjecture also predicts the leading term

$$L(E,s) = L^*(E) \cdot (s-1)^r + \cdots$$

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Birch and Swinnerton-Dyer conjecture

$$L^*(E) = \frac{\prod_p c_p \cdot \Omega \cdot \mathrm{Reg}(E/\mathbb{Q}) \cdot \#Ш(E/\mathbb{Q})}{\left(\#E(\mathbb{Q})_{\mathrm{tors}}\right)^2}$$

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- $Ш(E/\mathbb{Q})$ is the mysterious Tate-Shafarevich group.

The Tate-Shafarevich group

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- If it is then the parity $r_{\text{an}} \equiv r \pmod 2$ holds.
- If it is for a function field K then BSD is true for K.
- It is known to be finite for $\mathbb{Q}$ if and only if $r_{\text{an}} \leqslant 1$.

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- If $r_{\mathrm{an}} \leqslant 1$, the group $Ш(E/\mathbb{Q})$ is finite and the conjecture can be proven `sage: E.prove_bsd()`.
- Lots of numerical evidence for $r_{\mathrm{an}} \geqslant 2$.

$$\frac{L^*(E)}{\Omega \cdot \mathrm{Reg}(E/\mathbb{Q})} = \frac{\prod_p c_p \cdot \#Ш(E/\mathbb{Q})}{\left(\#E(\mathbb{Q})_{\mathrm{tors}}\right)^2}$$

$$E_2 \ : \ y^2 \ = \ x^3 \ + \ x \ + \ 2, \qquad r_{\mathrm{an}} = r = 0$$

- $L(E,1) \cong 0.874549$
- $\Omega \cong 3.49819$
- $\mathrm{Reg}(E/\mathbb{Q}) = 1$
- $L(E,1)/\Omega \cong 0.250000.$
- In fact $L(E,1)/\Omega = \frac{1}{4}$.
- $c_2 = 4$ and $c_p = 1 \, \forall_{p \neq 2}$.
- $\#E(\mathbb{Q}) = 4$
- $Ш(E/\mathbb{Q})$ is trivial.

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$$E_2 \ : \ y^2 \ = \ x^3 \ + \ x \ + \ 2, \qquad r_{\mathrm{an}} = r = 1$$

- $L'(E,1) \cong 1.78581$
- $\Omega \cong 3.74994$
- $\mathrm{Reg}(E/\mathbb{Q}) \cong 0.476223$
- $\mathrm{LHS} \cong 1.00000.$
- In fact it is 1.

- $c_p = 1.$
- $E(\mathbb{Q}) = \mathbb{Z}$
- $Ш(E/\mathbb{Q})$ is trivial.

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$$E_9 \ : \ y^2 \ = \ x^3 \ + \ x \ + \ 9, \qquad r_{\mathrm{an}} = r = 2$$

- $L^*(E) \cong 7.16561$
- $\Omega \cong 2.84721$
- $\mathrm{Reg}(E/\mathbb{Q}) \cong 2.51672$
- $\mathrm{LHS} \cong 1.00000.$
- $c_p = 1.$
- $E(\mathbb{Q}) = \mathbb{Z}^2$
- $Ш(E/\mathbb{Q})$ should be trivial.

$$\frac{L^*(E)}{\Omega \cdot \mathrm{Reg}(E/\mathbb{Q})} = \frac{\prod_p c_p \cdot \#Ш(E/\mathbb{Q})}{\left(\#E(\mathbb{Q})_{\mathsf{tors}}\right)^2}$$

$$E_{-47} \ : \ y^2 \ = \ x^3 \ + \ x \ - \ 47, \qquad r_{\mathsf{an}} = r = 0$$

- $L(E,1) \cong 5.15400$
- $\Omega \cong 1.28850$
- $\mathrm{Reg}(E/\mathbb{Q}) = 1$
- $L(E,1)/\Omega = 4.$

- $c_p = 1$
- $E(\mathbb{Q}) = 0$
- $Ш(E/\mathbb{Q}) = {}^{\mathbb{Z}}\!/_{2\mathbb{Z}} \oplus {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}.$

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Recent work of Skinner-Urban: If $Ш(E/\mathbb{Q})$ is finite and $\mathrm{Reg}_p(E/\mathbb{Q}) \neq 0$, then the p-adic BSD holds.

Equivariant version

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Often there is a formula for the leading term involving the $\mathbb{Z}[G]$-structure of $Ш(E/K)$.