Chromatic Polynomials of Plane Triangulations — Proofs

We will prove Theorems 3 and 4, and parts (b), (c) and (e) of Theorem 5.

Proof of Theorem 3. We use induction on $m+n$. There are three cases to consider.

Case 1: G is K_1 or K_2 or K_3. Then we have the following values.

G	c	b	γ	μ	$t^c(t-1)^b$	$q(G,t)$
K_1	1	0	0	0	t	1
K_2	1	1	0	0	$t(t-1)$	1
K_3	1	1	1	1	$t(t-1)$	$1+(1-t)$

The result clearly holds.

Case 2: $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \emptyset$ or K_1 or K_2, and $n_1 < n$ and $n_2 < n$. The values of various parameters are as follows.

$G_1 \cap G_2$	m_1+m_2	n_1+n_2	c_1+c_2	b_1+b_2
$\emptyset$	m	n	c	b
K_1	m	$n+1$	$c+1$	b
K_2	$m+1$	$n+2$	$c+1$	$b+1$

Note that $\gamma_1+\gamma_2=\gamma$ and $\mu_1+\mu_2=\mu$ in each case. Since

$$P(G,t) = \frac{P(G_1,t)P(G_2,t)}{t^{c_1+c_2-c}(t-1)^{b_1+b_2-b}}$$

by Proposition 2 (b) and the values in the above table, it follows that

$$q(G,t) = q(G_1,t)\,q(G_2,t).$$

We may suppose inductively that the result holds for G_1 and G_2. Thus $q(G,t)$ is a polynomial in t and

$$\begin{aligned} q(G,t) &\gg_s (1+\gamma_1 s+\gamma_1 s^2+\ldots+\gamma_1 s^{\mu_1-1}+s^{\mu_1})\times \\ &\qquad\qquad \times(1+\gamma_2 s+\gamma_2 s^2+\ldots+\gamma_2 s^{\mu_2-1}+s^{\mu_2}) \\ &\gg_s 1+\gamma s+\gamma s^2+\ldots+\gamma s^{\mu-1}+s^{\mu} \end{aligned}$$

as required. (Note that if $\mu \leqslant 1$ then G is a forest or contains just one triangle, and $\gamma = \mu$.)

Case 3: G is 2-connected, $\neq K_3$, and if $e \in E(G)$ then G/e is 2-connected. If $e \in E(G)$ let $G_1 := G - e$ and $G_2 := G/e$ (without parallel edges). Then $n_1 = n$, $n_2 = n-1$ and $c_1 = c_2 = c = b_2 = b = 1$, so that $\mu - \mu_1 = b_1 - 1$ and $\mu_2 = \mu - 1$. We may suppose inductively that the result holds for G_1 and G_2. The deletion-contraction formula gives

$$q(G,t) = (1-t)^{b_1-1} q(G_1,t) + q(G_2,t),$$

which is a polynomial in t. Also, since $\gamma_1 = \gamma - 1$,

$$\begin{aligned} (1-t)^{b_1-1} q(G_1,t) &\gg_s s^{\mu-\mu_1}[1+(\gamma-1)s+\ldots+(\gamma-1)s^{\mu_1-1}+s^{\mu_1}] \\ &\gg_s (\gamma-1)s^{\mu-1}+s^{\mu}. \end{aligned}$$

If e does not lie in a triangle then $\gamma_2 = \gamma$. Since $\mu_2 = \mu - 1$,

$$q(G_2,t) \gg_s 1+\gamma s+\gamma s^2+\ldots+\gamma s^{\mu-2}+s^{\mu-1}$$

and the result follows. But if *every* edge lies in a triangle, and $G \neq K_3$, it is not difficult to find an edge e such that $G_1 = G - e$ is 2-connected, so that $b_1 = 1$ and $\mu_1 = \mu$. Since G_2 is 2-connected, $\gamma_2 \geqslant 1$ and

$$q(G_2,t) \gg_s 1+s+s^2+\ldots+s^{\mu-1},$$

whence the result follows. //

Proof of Theorem 4. We use induction on m. There are three cases to consider.

Case 1: Some vertex v of F has degree 2. If $G = K_3$ then $n = k = 3$, $m' = 0$, $q(G, t) = 1$ and the result is obvious. Otherwise, $G - v$ is a near-triangulation with $(k-1)$-face, $P(G, t) = (t-2)P(G - v, t)$, and, by induction,

$$\begin{aligned} q(G,t) &= (2-t)q(G-v,t) \\ &\gg_r r \times r^{(k-1)-3}(1+r)^{(n-1)-(k-1)-m'} \\ &= r^{k-3}(1+r)^{n-k-m'}, \end{aligned}$$

as required.

Case 2: G contains a separating C_2 or C_3. Let G_1 (resp. G_2) consist of this circuit and everything outside it (resp. inside it), but delete one edge of C_2 in each case. Some values are:

circuit	$n_1 + n_2$	$m'_1 + m'_2$
C_2	$n + 2$	$m' - 1$
C_3	$n + 3$	m'

Suppose w.l.o.g. F is a face of G_1. Then G_1 is a near-triangulation with k-face F and G_2 is a triangulation. We may suppose inductively that the result holds for these graphs. Since

$$P(G,t) = \frac{P(G_1,t)P(G_2,t)}{t(t-1)(t-2)^{m'_1+m'_2+1-m'}}$$

by Proposition 2 (b), it follows that

$$q(G,t) = q(G_1,t)\, q(G_2,t),$$

which is a polynomial in t. Moreover

$$\begin{aligned} q(G,t) &\gg_r r^{k-3}(1+r)^{n_1-k-m'_1} r^{3-3}(1+r)^{n_2-3-m'_2} \\ &= r^{k-3}(1+r)^{n-k-m'} \end{aligned}$$

as required.

Case 3: G is simple ($m' = 0$) and has no separating triangle, and no vertex of F has degree 2. It is not difficult to see that $n > k$ and there is a triangular face of G with exactly one edge e in F and one vertex v not in F. Then $G - e$ is a simple near-triangulation with $(k+1)$-face, and G/e becomes simple and a near-triangulation with $(k-1)$-face (if $k \geqslant 4$) or a triangulation (if $k = 3$) after the deletion of one or two edges, respectively, from pairs of parallel edges. We may suppose inductively that the result holds for $G - e$ and G/e. By the deletion-contraction formula,

$$q(G,t) = q(G-e,t) + q(G/e,t),$$

which is thus a polynomial in t. By induction,

$$q(G-e,t) \gg_r r^{(k+1)-3}(1+r)^{n-(k+1)}$$

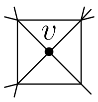
and

$$q(G/e,t) \gg_r \begin{cases} r^{(k-1)-3}(1+r)^{(n-1)-(k-1)} & \text{if } k \geqslant 4, \\ r^{3-3}(1+r)^{(n-1)-3} & \text{if } k = 3, \end{cases}$$
$$\gg_r r^{k-3}(1+r)^{n-1-k}.$$

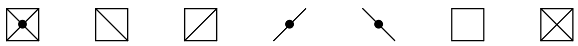
Thus

$$q(G,t) \gg_r (r+1)r^{k-3}(1+r)^{n-1-k} = r^{k-3}(1+r)^{n-k}$$

as required. //

Lemmas for Theorem 5. Let G be a loopless plane graph containing the configuration 

. The symbols



denote the chromatic polynomials of G and the graphs obtained from G in the ‘obvious’ way (the last graph being probably nonplanar).

For example, $\square = P(G - v, t)$, etc. Let $\Sigma_1 := \boxbslash + \boxslash$ and $\Sigma_2 := \dot\diagup + \dot\diagdown$.

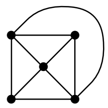
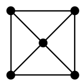
Lemma 5.1. (a) $\dot\boxtimes = \frac{1}{2}(t-4)\Sigma_1 + \frac{1}{2}(t-2)\Sigma_2$.

(b) $(t-1)\dot\boxtimes = (t-2)^2\square - \frac{t^2-3t+1}{t-3}\boxtimes$.

Proof. (a)

$\dot\boxtimes = \dot\boxtimes_3 - \boxbslash$	by deletion-contraction,
$= \dot\boxtimes_2 - \boxslash - \boxbslash$	by deletion-contraction,
$= (t-2)\square - \boxslash - \boxbslash$	
$= (t-3)\boxslash + (t-2)\dot\diagdown - \boxbslash$	by addition-identification,
$= \frac{1}{2}(t-4)\Sigma_1 + \frac{1}{2}(t-2)\Sigma_2$	symmetrizing.

(b) We use induction on $|V(G)| + |E(G)|$. There are only two essentially different such graphs with five vertices, for which it is easy to check the result from the following table (in which each entry is the relevant polynomial with a factor $t(t-1)$ removed).

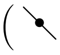
graph / polynomial		
$\dot\boxtimes$	$(t-2)(t-3)^2$	$(t-2)[(t-3)^2 + (t-2)]$
$\square$	$(t-2)^2$	$(t-2)^2 + (t-1)$
$\boxtimes$	$(t-2)(t-3)$	$(t-2)(t-3)$

For larger graphs, remove any isolated vertex or apply deletion-contraction to any 'additional' edge to prove the result by induction. //

Let $\tau := \frac{1}{2}(1+\sqrt{5})$, so that $\tau^2 - \tau - 1 = 0$, $\tau^2 = \tau + 1$, $\tau - 1 = \tau^{-1}$, and $2 - \tau = \tau^2 - 2\tau + 1 = (\tau-1)^2 = \tau^{-2}$.

Lemma 5.2. (a) $\boxbslash - \boxslash = \dot\diagdown - \dot\diagup$.

(b) At $\tau+1$ (i.e., evaluating all polynomials at $t=\tau+1$),

 $^2 -$  $^2 = \tau^{-3}($  $^2 -$  $^2)$.

Proof. (a) follows immediately from the fact that

 $=$  $+$  $=$  $+$  $= \frac{1}{2}(\Sigma_1+\Sigma_2)$ (symmetrizing).

For (b), let $t=\tau+1$, so that $t^2-3t+1=0$. Then Lemma 5.1 (b) gives

τ  $= \tau^{-2}$ 

$$= \tfrac{1}{2}\tau^{-2}(\Sigma_1+\Sigma_2) = \tfrac{1}{2}(2-\tau)(\Sigma_1+\Sigma_2).$$

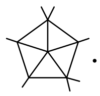
Lemma 5.1 (a) gives

 $= \frac{1}{2}(\tau-3)\Sigma_1 + \frac{1}{2}(\tau-1)\Sigma_2$.

Eliminating ,

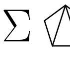
$$\begin{aligned} 0 &= (\tau^2-3\tau-2+\tau)\Sigma_1 + (\tau^2-\tau-2+\tau)\Sigma_2 \\ &= -\tau^2\Sigma_1 + \tau^{-1}\Sigma_2, \end{aligned}$$

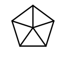
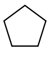
so that $\Sigma_1 = \tau^{-3}\Sigma_2$, and (b) follows immediately from this and (a). //

Let G now be a loopless plane graph containing the configuration . Symbols such as

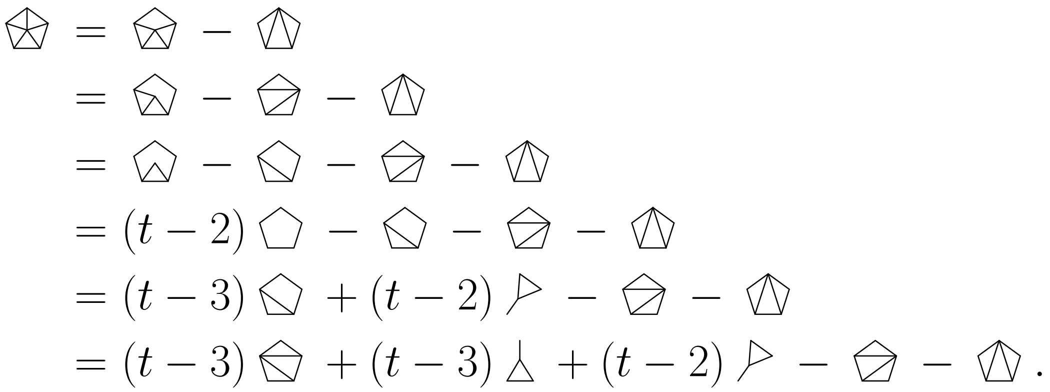
will be used in a way that is hopefully self-explanatory. Σ will denote the sum over the five rotations, so that, for example,

Σ  $:=$  $+$  $+$  $+$  $+$  .

Lemma 5.3. (a)  $= \frac{1}{5}(t-5)\Sigma$  $+ \frac{1}{5}(2t-5)\Sigma$  .

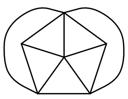
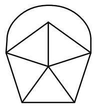
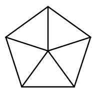
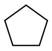
(b) $(t-1)$  $= (t-2)(t-3)$  $- \frac{t^2-3t+1}{t-4}$  .

Proof. (a)



Symmetrizing, we obtain the result.

(b) We use induction on $|V(G)|+|E(G)|$. There are three essentially different such graphs with six vertices, for which it is easy to check the result from the following table (in which each entry is the relevant polynomial with a factor $t(t-1)(t-2)$ removed).

graph / polynomial			
	$(t-3)^3$	$(t-3)[(t-3)^2+(t-2)]$	$(t-3)[(t-3)^2+2(t-2)]$
	$(t-2)^2$	$(t-2)^2+(t-1)$	$(t-2)^2+2(t-1)$
	$(t-3)(t-4)$	$(t-3)(t-4)$	$(t-3)(t-4)$

The result follows exactly as in Lemma 5.1 (b). //

Theorem 5. Let G be a plane triangulation with n vertices and let $s := t-5$.

(b) $|P(G,\tau+1)| \leqslant \tau^{5-n}$.

(c) $P(G,\tau+2) = (\tau+2)\tau^{3n-10}P(G,\tau+1)^2$.

(e) If $t \geqslant 5$ then $P(G,t)$ and all its derivatives (up to the nth) are > 0.

(f) $P(G,t) = (s+5)(s+4)(s+3)q(G,s)$, where $q(G,s)$ is a polynomial with all coefficients > 0 (from s^0 up to s^{n-3}).

Proof. (e) says that if $P(G,t)$ is rewritten as a polynomial in s, then all its coefficients are > 0. This follows from (f). We will prove (f), (b) and (c) by induction on n, noting that they all hold if $n = 3$, when $G = K_3$,

$$P(G,t) = t(t-1)(t-2) = (s+5)(s+4)(s+3),$$
$$P(G,\tau+1) = (\tau+1)\tau(\tau-1) = \tau^2\tau\tau^{-1} = \tau^{5-3}$$

and

$$P(G,\tau{+}2) = (\tau{+}2)(\tau{+}1)\tau = (\tau{+}2)\tau^3 = (\tau{+}2)\tau^{3\cdot3-10}P(G,\tau{+}1)^2.$$

So suppose $n \geqslant 4$. There are three cases.

Case 1: G contains a separating C_2 or C_3. Then $G = G_1 \cup G_2$ where $G_1 \cap G_2 = C_2$ or C_3, $n_1 + n_2 = n+2$ or $n+3$, $n_1 < n$, $n_2 < n$, and

$$P(G,t) = \frac{P(G_1,t)P(G_2,t)}{t(t-1)(t-2)^{n_1+n_2-n-2}}$$

by Proposition 2 (b). We may suppose inductively that (f), (b) and (c) hold for G_1 and G_2. Therefore

$$P(G,t) = \frac{(s+5)^2(s+4)^2(s+3)^2 q(G_1,s)q(G_2,s)}{(s+5)(s+4)(s+3)^{n_1+n_2-n-2}},$$
$$q(G,s) = (s+3)^{n+3-n_1-n_2}q(G_1,s)q(G_2,s),$$

and so (f) holds for G. Also

$$|P(G,\tau+1)| \leqslant \frac{\tau^{5-n_1}\tau^{5-n_2}}{(\tau{+}1)\tau(\tau{-}1)^{n_1+n_2-n-2}} = \frac{\tau^{10-n-2}}{\tau^2\tau} = \tau^{5-n},$$

so that (b) holds. And $P(G,\tau+2) =$

$$= \frac{(\tau{+}2)\tau^{3n_1-10}P(G_1,\tau{+}1)^2\,(\tau{+}2)\tau^{3n_2-10}P(G_2,\tau{+}1)^2}{(\tau{+}2)(\tau{+}1)\tau^{n_1+n_2-n-2}}$$

$$= (\tau{+}2)\tau^{3n-10}\frac{\tau^{-4}}{\tau{+}1}\tau^{2(n_1+n_2-n-2)}P(G_1,\tau{+}1)^2P(G_2,\tau{+}1)^2$$

$$= (\tau{+}2)\tau^{3n-10}P(G,\tau{+}1)^2$$

since $\frac{\tau^{-4}}{\tau+1}\tau^{2(n_1+n_2-n-2)} = \frac{1}{(\tau+1)^2\tau^2(\tau-1)^{2(n_1+n_2-n-2)}}$. So (c) holds.

Case 2: G is simple and contains a vertex v of degree 4. Then the neighbours of v are all distinct, and we may apply Lemmas 5.1 and 5.2. Lemma 5.1 (a) gives

$$\begin{aligned} P(G,t) = \boxtimes &= \tfrac{1}{2}(t-4)(\boxbslash + \boxslash) + \tfrac{1}{2}(t-2)(\diagup + \diagdown) \\ &= \tfrac{1}{2}(s+1)(\boxbslash + \boxslash) + \tfrac{1}{2}(s+3)(\diagup + \diagdown), \end{aligned}$$

and (f) follows by induction. Evaluating all polynomials at $\tau+1$, Lemma 5.1 (b) gives

$$\begin{aligned} \tau\boxtimes &= (\tau-1)^2\square \qquad \text{since } t^2-3t+1=0 \\ &= \tau^{-2}(\boxbslash + \diagup). \end{aligned}$$

We may suppose inductively that

$$|\boxbslash| \leqslant \tau^{5-(n-1)} \quad \text{and} \quad |\diagup| \leqslant \tau^{5-(n-2)},$$

so that

$$|\boxtimes| \leqslant \tau^{-3}(\tau^{5-n+1} + \tau^{5-n+2}) = \tau^{-3}(\tau+\tau^2)\tau^{5-n} = \tau^{5-n}.$$

Thus (b) holds. Finally, Lemma 5.2 (a) gives

$$\boxtimes - \boxbslash = \varnothing - \boxslash.$$

We may suppose inductively that $\boxbslash$, $\varnothing$ and $\boxslash$ all satisfy (c) (since $\boxbslash$ can immediately be reduced by Case 1), and then

$$\begin{aligned} P(G,\tau+2) &= [\boxbslash + \varnothing - \boxslash]_{\tau+2} \\ &= (\tau+2)\tau^{3n-10}\left[\boxbslash^2 + \tau^{-3}\left(\varnothing^2 - \boxslash^2\right)\right]_{\tau+1} \\ &= (\tau+2)\tau^{3n-10}\left[\boxbslash^2 + \boxtimes^2 - \boxbslash^2\right]_{\tau+1} \qquad \text{by Lemma 5.2 (b)} \\ &= (\tau+2)\tau^{3n-10}P(G,\tau+1)^2. \end{aligned}$$

So (c) holds.

Case 3: G is simple and contains a vertex v of degree 5. Lemma 5.3 (a) gives

$$P(G,t) = \quad = \tfrac{1}{5}(t-5)\sum \quad + \tfrac{1}{5}(2t-5)\sum$$
$$= \tfrac{1}{5}s\sum \quad + \tfrac{1}{5}(2s+5)\sum \quad ,$$

and (f) follows by induction. Evaluating all polynomials at $\tau+1$, Lemma 5.3 (b) gives

$$\tau \quad = (\tau-1)(\tau-2)$$
$$= -\tau^{-1}\tau^{-2}\,(\quad + \quad)$$
$$= -\tau^{-3}\,(\quad + \quad + \quad),$$

so that

$$|\quad| \leqslant \tau^{-4}\left(\tau^{5-n+1} + 2\tau^{5-n+2}\right) = \tau^{-4}(\tau+2\tau^2)\tau^{5-n} = \tau^{5-n}$$

since $1+2\tau = \tau+\tau^2 = \tau^3$. Thus (b) holds. Finally, Lemma 5.2 (a) gives

$$\quad - \quad = \quad - \quad .$$

We may suppose inductively that the last three graphs all satisfy (c) (since can immediately be reduced by Case 2), and then

$$P(G,\tau+2) = [\quad + \quad - \quad]_{\tau+2}$$
$$= (\tau+2)\tau^{3n-10}\left[\quad^2 + \tau^{-3}\left(\quad^2 - \quad^2\right)\right]_{\tau+1}$$
$$= (\tau+2)\tau^{3n-10}\left[\quad^2 + \quad^2 - \quad^2\right]_{\tau+1} \quad \text{by Lemma 5.2 (b)}$$
$$= (\tau+2)\tau^{3n-10}P(G,\tau+1)^2.$$

So (c) holds. //