Edge-Colourings of Simple Graphs

If a graph G is properly edge-coloured, let $C(v)$ denote the set of colours present on edges at vertex v. An (a, b)-*chain* is a component (a maximal path or a circuit) of the subgraph of edges coloured a or b.

Theorem 1. (D. König, 1916.) If G is a bipartite multigraph then $\chi'(G) = \Delta(G)$.

Proof. Suppose $\chi'(G) \geqslant \Delta(G) + 1$ and let $t := \Delta(G)$. Let H be a minimal subgraph of G that is not edge-t-colourable, and let $e = uw \in E(H)$. Choose an edge-t-colouring of $H - e$. Every colour c is used at u or w, otherwise we could colour e with c. But there are colours $c_u \notin C(u)$ and $c_w \notin C(w)$, since $d_{H-e}(u) < t$ and $d_{H-e}(w) < t$. Let P be the (c_w, c_u)-chain starting from u. Since H is bipartite, P could only reach u or w along an edge coloured c_u or c_w respectively, which is impossible. So interchange the colours c_u and c_w along P and colour e with c_w. This shows that $\chi'(H) \leqslant t$, a contradiction. //

Vizing proved a general upper bound for $\chi'(G)$, which we will prove only for simple graphs. From now on, assume G is simple. A *fan sequence* at a vertex w is a sequence of distinct edges of the form $wv_1, \ldots, wv_s$ such that, for each $i \geqslant 2$, $c(wv_i) \notin C(v_{i-1})$.

Lemma 2.1. Suppose $t \geqslant \Delta(G)$ and G is not edge-t-colourable but $G - e$ is, for some edge $e = uw$ of G. For some edge-t-colouring c of $G - e$, suppose there is an edge wv_1 s.t. $c(wv_1) \notin C(u)$, and let $wv_1, \ldots, wv_s$ be a longest fan sequence starting with wv_1. Then

(i) $\forall\, i$, $C(u) \setminus C(w) \subseteq C(v_i)$;
(ii) if $j < i$ then $c(wv_j) \in C(v_i)$;
(iii) $d(v_s) = \Delta(G) = t$.

Proof. If $a \in C(u) \setminus C(w)$ and $a \notin C(v_i)$ then we can (re)colour

$$\begin{array}{rccccc} & e, & wv_1, & \ldots, & wv_{i-1}, & wv_i \\ \text{with} & c(wv_1), & c(wv_2), & \ldots, & c(wv_i), & a. \end{array} \qquad (*)$$

This contradiction proves (i). To prove (ii), note that $C(u) \setminus C(w) \neq \emptyset$, since clearly $|C(u) \cup C(w)| = t$ and so $C(u) \subseteq C(w)$ would imply $d(w) \geqslant 1 + t > \Delta(G)$. Fix $a \in C(u) \setminus C(w)$, and suppose $\exists\, j < i$ s.t. $c(wv_j) \notin C(v_i)$. Let P be the $(a, c(wv_j))$-chain starting from v_i.

Case 1: $j = 1$. If P does not end at u (even if P ends along $v_1 w$) interchange colours along P and then use $(*)$. If P does end at u then interchange colours along the $(a, c(wv_1))$-chain starting along wv_1 (which does not end at u) and colour e with $c(wv_1)$. Either way we have a contradiction.

Case 2: $j \geqslant 2$. If P does not end at v_{j-1}, interchange colours along P and then use $(*)$. Otherwise, interchange colours along P and then (re)colour

$$\begin{array}{rccccc} & e, & wv_1, & \ldots, & wv_{j-2}, & wv_{j-1} \\ \text{with} & c(wv_1), & c(wv_2), & \ldots, & c(wv_{j-1}), & a. \end{array}$$

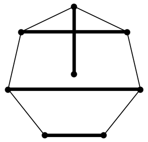
This contradiction proves (ii). Now (iii) follows because if any colour b were missing from v_s, then by (i) and (ii), and since $|C(u) \cup C(w)| = t$, there would be an edge at w of colour $b \notin \{c(wv_1), \ldots, c(wv_s)\}$, and so there would be a longer fan sequence. Thus all t colours are present at v_s. //

Theorem 2. *Vizing's Theorem* (V. G. Vizing, 1964; R. P. Gupta, 1966.) For every simple graph G with maximum degree Δ, $\chi'(G) \leqslant \Delta + 1$.

Proof. Suppose $\chi'(G) \geqslant \Delta + 2$. Set $t = \Delta + 1$ and let H be a minimal subgraph of G that is not edge-t-colourable. Then $H - e$

is edge-t-colourable, for each edge $e = uw$ of H, and so Lemma 2.1 implies that $t = \Delta(H) \leqslant \Delta$, a contradiction. //

It follows from Vizing's theorem that $\chi'(G) = \Delta(G)$ or $\Delta(G)+1$ for every simple graph G. G is said to be of *class one* if $\chi'(G) = \Delta(G)$ and of *class two* if $\chi'(G) = \Delta(G)+1$. Theorem 1 shows that every bipartite graph is of class one. So is K_n (n even):



times $n-1$ rotations

But K_n (n odd) is of class two, since each colour can be used on at most $\lfloor \frac{1}{2}n \rfloor = \frac{1}{2}(n-1)$ edges and so $\chi'(K_n) \geqslant \frac{\frac{1}{2}n(n-1)}{\frac{1}{2}(n-1)} = n = \Delta(K_n)+1$. A graph G is called *overfull* if $|E(G)| > \Delta(G)\lfloor \frac{1}{2}|V(G)| \rfloor$. If G is overfull then $|V(G)|$ is odd and G is of class two.

Conjecture. (A. G. Chetwynd and A. J. W. Hilton, 1986.) If $\Delta(G) > \frac{1}{3}|V(G)|$, then G is of class two if and only if G contains an overfull subgraph H with $\Delta(H) = \Delta(G)$.

The Petersen graph minus 1 vertex is of class two, and has no such overfull subgraph, and has $\Delta = \frac{1}{3}|V|$.

Lemma 3.1. Let G and c be as in Lemma 2.1 with $t = \Delta(G)$. Let $wv_1, \ldots, wv_s$ and $wx_1, \ldots, wx_r$ be fan sequences at w with $v_1 \neq x_1$ and $c(wv_1), c(wx_1) \notin C(u)$. Then $v_i \neq x_j$, $\forall\, i, j$.

Proof. Suppose $v_i = x_j$ and (w.l.o.g.) $i \geqslant 2$ and the vertices $v_1, \ldots, v_i, x_1, \ldots, x_j$ are otherwise distinct. Let $a \in C(u) \setminus C(w)$ as before, and interchange colours along the $(c(wv_i), a)$-chain P starting along $wv_i = wx_j$. If P does not end at v_{i-1} we can use $(*)$. So suppose P ends at v_{i-1}. If $j = 1$, we can simply colour e with $c(wv_i)$, since $c(wv_i) = c(wx_1) \notin C(u)$ and P does not end at u. If $j \geqslant 2$, we can reverse the roles of the two fans and then use $(*)$ (since P does not end at x_{j-1}). //

Theorem 3. *Vizing's Adjacency Lemma* (1965). Let G be a minimal graph that is not edge-Δ-colourable, where $\Delta = \Delta(G)$, and let u, w be adjacent vertices of G, where $d(u) = k$. Then w is adjacent to at least $\Delta - k + 1$ vertices of degree Δ different from u.

Proof. Consider an edge-Δ-colouring of $G - uw$. For each of the $\Delta - k + 1$ colours appearing at w but not at u, there is a longest fan sequence at w starting with an edge of that colour. By Lemma 3.1 these fan sequences are all disjoint, and by Lemma 2.1 each ends with an edge wv_s with $d(v_s) = \Delta$. //

Corollary 3.1. Each vertex of G has at least two neighbours of degree Δ, and G contains at least three Δ-vertices. //

Theorem 4. (V. G. Vizing, 1968.) If G is a simple planar graph with maximum degree $\Delta \geqslant 8$, then $\chi'(G) = \Delta$.

Proof. (H. Hind and Y. Zhao, DM 190 (1998) 107–114.) Let $G = (V, E, F)$ be a minimal counterexample. Then $d(u) \geqslant 2$ and $d(u) + d(w) \geqslant \Delta + 2 \geqslant 10$ for each edge uw. Let $d(f)$ denote the number of edges bounding face f. Euler's formula $|V| - |E| + |F| \geqslant 2$ gives

$$\sum_{v \in V}(4 - d(v)) + \sum_{f \in F}(4 - d(f)) \geqslant 8. \tag{1}$$

Assign 'charge' $M(x) := 4 - d(x)$ to each $x \in V \cup F$. Note that only 2-vertices, 3-vertices and 3-faces have positive charge, but the total charge is positive by (1). Now redistribute the charge as follows.

For each edge uw,

(R1) if $d(u) = 2$ then $d(w) = \Delta$: send 1 from u to w;

(R2) if $d(u) = 3$ then $d(w) \geqslant \Delta - 1$: send $\frac{1}{3}$ from u to w.

For each 3-face $f = uvw$,

(R3) if $d(u) \leqslant 4$ and $d(v), d(w) \geqslant 7$, f sends $\frac{1}{2}$ to each of v, w: $\frac{1}{3}$ directly and $\frac{1}{6}$ via u;

(R4) if $d(u) = 4$ and $d(v) = 6$, then $d(w) = \Delta = 8$, since $d(u) + d(v) \geqslant \Delta + 2$, and if $d(u), d(w) < \Delta$ then $d(v) \geqslant 2 + (\Delta - d(u) + 1) \geqslant 7$ by Theorem 3; and f sends $\frac{1}{3}$ to v and $\frac{2}{3}$ to w: $\frac{1}{3}$ directly and $\frac{1}{6}$ via each of u, v;

(R5) if $d(u), d(v), d(w) \geqslant 5$ then f sends $\frac{1}{3}$ to each of u, v, w.

Finally,

(R6) each 5-vertex v sends $2/(3d_{\geqslant 7}(v))$ to each vertex in $N_{\geqslant 7}(v)$, where $d_i(v)$ and $N_i(v)$ denote the number and set of neighbours of v with degree i, etc. Note that $d_{\geqslant 7}(v) \geqslant 2$ by Corollary 3.1, and v loses exactly $\frac{2}{3}$ by (R6).

Claim. If $d(w) \geqslant 7$, then the charge that w receives by (R6):
(a) from each neighbouring 5-vertex is $\leqslant \frac{1}{6}$ if $d(w) = 7$, $\leqslant \frac{2}{9}$ otherwise;
(b) in total, is $\leqslant \frac{1}{2}$ if $d(w) = 7$, $\leqslant \frac{8}{9}$ otherwise.

Proof of Claim. Suppose $v \in N_5(w)$. If $d_{\leqslant 6}(v) \geqslant 1$, then $d_\Delta(v) \geqslant \Delta - 6 + 1 \geqslant 3$ by Theorem 3. Thus $d_{\geqslant 7}(v) \geqslant 3$, and $\geqslant 4$ if $d(w) < \Delta$. This proves (a). (b) follows because if $d_5(w) > 0$ then $d_\Delta(w) \geqslant \Delta - 4$ by Theorem 3, so that $d_5(w) \leqslant 4$, and $\leqslant 3$ if $d(w) < \Delta$. This proves the Claim. //

If $x \in V \cup F$, let the charge on x after the redistribution be $M'(x)$, $= M(x) + R(x)$, where $R(x)$ is the net change. If x is a 3-face then $M(x) = 1$, $R(x) = -1$ by (R3)–(R5), and so $M'(x) = 0$. If x is any other face then $M'(x) = M(x) \leqslant 0$. So suppose $x = v$, a vertex.

$\underline{d(v) = 2}$: $M(v) = 2$, $R(v) = -2$ by (R1), $M'(v) = 0$.
$\underline{d(v) = 3}$: $M(v) = 1$, $R(v) = -1$ by (R2), $M'(v) = 0$.
$\underline{d(v) = 4}$: $M(v) = 0$, $R(v) = 0$, $M'(v) = 0$.
$\underline{d(v) = 5}$: $M(v) = -1$, $R(v) \leqslant \frac{5}{3} - \frac{2}{3}$ by (R5) & (R6), $M'(v) \leqslant 0$.
$\underline{d(v) = 6}$: $M(v) = -2$, $R(v) \leqslant \frac{6}{3}$ by (R4) & (R5), $M'(v) \leqslant 0$.
$\underline{d(v) = 7}$: $M(v) = -3$, so we must prove $R(v) \leqslant 3$. Now,

$$R(v) \leqslant \underbrace{\tfrac{7}{3}}_{(\mathrm{R3,R4,R5})} + \underbrace{\tfrac{2d_3(v)}{3}}_{(\mathrm{R2,R3})} + \underbrace{\tfrac{d_4(v)}{3}}_{(\mathrm{R3})} + \underbrace{\tfrac{d_5(v)}{6}}_{(\mathrm{R6})=(\mathrm{a})}.$$

If $d_3(v) > 0$ then $d_\Delta(v) = 6$ by Theorem 3, so $R(v) \leqslant \frac{7}{3} + \frac{2}{3} = 3$.
If $d_3(v) = 0$ but $d_4(v) > 0$ then $d_\Delta(v) \geqslant 5$ by Theorem 3, so $d_4(v) + d_5(v) \leqslant 2$; again $R(v) \leqslant \frac{7}{3} + \frac{2}{3} = 3$.
If $d_{\leqslant 4}(v) = 0$ then $R(v) \leqslant \frac{7}{3} + \frac{1}{2} < 3$ by (b) of the Claim.

$\underline{d(v) = k \geqslant 8}$: $M(v) = 4 - k$ and

$$R(v) \leqslant \underbrace{\tfrac{k}{3}}_{(\mathrm{R3,R4,R5})} + \underbrace{\tfrac{5d_2(v)}{6}}_{(\mathrm{R1,R3})} + \underbrace{\tfrac{2d_3(v)}{3}}_{(\mathrm{R2,R3})} + \underbrace{\tfrac{d_4(v)}{3}}_{(\mathrm{R3,R4})} + \underbrace{\tfrac{2d_5(v)}{9}}_{(\mathrm{R6})=(\mathrm{a})} + \underbrace{\tfrac{\min\{d_4(v), d_6(v)\}}{6}}_{(\mathrm{R4})}$$

since (i) if $w \in N_2(v)$ then, since G is simple, vw lies in the boundary of at most one 3-face, and so v receives 1 from w by (R1) and $\leqslant \frac{1}{2}$ from the two faces either side of edge vw by (R3), instead of the $\frac{2}{3}$ allowed for in the first term; and (ii) a 4-vertex is adjacent to at most one 6-vertex, and a 6-vertex to at most one 4-vertex, by Theorem 3. It suffices to prove that $R(v) \leqslant \frac{k}{3} + \frac{4}{3}$, since $4 - k + \frac{k}{3} + \frac{4}{3} = \frac{2}{3}(8 - k) \leqslant 0$.

If $d_2(v) > 0$ then $d_\Delta(v) = \Delta - 1$ and $R(v) \leqslant \frac{k}{3} + \frac{5}{6} < \frac{k}{3} + \frac{4}{3}$.
If $d_2(v) = 0$ but $d_3(v) > 0$ then $d_\Delta(v) \geqslant \Delta - 2$ and $R(v) \leqslant \frac{k}{3} + \frac{4}{3}$.
If $d_{\leqslant 3}(v) = 0$ but $d_4(v) > 0$ then $d_\Delta(v) \geqslant \Delta - 3$ and
$$R(v) \leqslant \frac{k}{3} + \frac{3}{3} < \frac{k}{3} + \frac{4}{3}.$$
If $d_{\leqslant 4}(v) = 0$ then $d_\Delta(v) \geqslant \Delta - 4$ and $R(v) \leqslant \frac{k}{3} + \frac{8}{9} < \frac{k}{3} + \frac{4}{3}$.

Thus $M'(x) \leqslant 0 \ \ \forall x \in V \cup F$, and this contradicts (1). //

Vizing conjectured the result of Theorem 4 if $\Delta \geqslant 6$. (It is false if $2 \leqslant \Delta \leqslant 5$: subdivide an edge of C_4, K_4, octahedron, icosahedron.) L. Zhang (Graphs Combin. 16 (2000) 467–495) and D. P. Sanders and Y. Zhao (JCT(B) 83 (2001) 201–212) proved it if $\Delta = 7$. Both this and the proof of Theorem 4 work also in the projective plane. For the torus and Klein bottle, the result is false if $\Delta = 6$ (consider $K_7 - e$); it was proved by Hind and Zhao (loc. cit.) for $\Delta \geqslant 8$, and by Sanders and Zhao (JCT(B) 87 (2003) 254–263) for $\Delta = 7$.