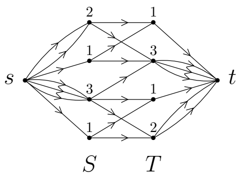
Matchings and Factors of Graphs

1. Necessary and sufficient conditions. Let G be a multigraph and $f : V(G) \to \mathbb{N} \cup \{0\}$ be a function. An *f-factor* of G is a subgraph H such that $d_H(v) = f(v)\ \forall\, v \in V(G)$. A *$k$-factor* is an f-factor s.t. $f(v) = k\ \forall\, v \in V(G)$. A *matching* is a subgraph with maximum degree $\leqslant 1$; i.e., a set of pairwise nonadjacent edges. A *perfect matching* is a 1-factor.

Theorem 1. Let G be a bipartite multigraph on two sets S, T of vertices. Then G has an f-factor if and only if

(i) $\sum_{v\in S} f(v) = \sum_{w\in T} f(w), = \Sigma$, say; and

(ii) $\forall\, X \subseteq S,\ Y \subseteq T$, the number $e(X, Y)$ of edges of G between X and Y is at least $\sum_{v\in X} f(v) + \sum_{w\in Y} f(w) - \Sigma$.



Proof. Condition (i) is clearly necessary; so suppose it holds. Form $\vec{G}$ by directing all edges of G from S to T, and adding a new vertex s joined to each v in S by $f(v)$ arcs, and a new vertex t with $f(w)$ arcs from each w in T to t. Then G has an f-factor if and only if $\vec{G}$ contains Σ arc-disjoint s, t-paths. The edge-separation analogue of Menger's theorem says that this happens if and only if every set of arcs separating s from t in $\vec{G}$ contains at least Σ arcs. Suppose C is a separating set of arcs. Let

$$\begin{aligned} X &:= \{v \in S : \text{not all } sv\text{-arcs are in } C\}, \\ Y &:= \{w \in T : \text{not all } wt\text{-arcs are in } C\}. \end{aligned}$$

Then all XY-arcs are in C, and so

$$\begin{aligned} |C| &\geqslant \textstyle\sum_{v\in S\setminus X} f(v) + \sum_{w\in T\setminus Y} f(w) + e(X, Y) \\ &= 2\Sigma - \textstyle\sum_{v\in X} f(v) - \sum_{w\in Y} f(w) + e(X, Y). \end{aligned}$$

Moreover, $\forall\, X \subseteq S,\ Y \subseteq T$, there exists a separating set with this number of arcs. So G has an f-factor if and only if, $\forall\, X \subseteq S$, $Y \subseteq T$,

$$2\Sigma - \textstyle\sum_{v\in X} f(v) - \sum_{w\in Y} f(w) + e(X, Y) \geqslant \Sigma,$$

which is (ii). //

Corollary 1.1. G has a 1-factor if and only if $|N(X)| \geqslant |X|$ $\forall\, X \subseteq V(G)$. (Here $N(X) := \bigcup_{x\in X} N(x)$.)

Proof. The condition is clearly necessary. To prove that it is sufficient, note first that it implies

$$|S| \geqslant |N(T)| \geqslant |T| \quad \text{and} \quad |T| \geqslant |N(S)| \geqslant |S|,$$

so that $|S| = |T|$, which is (i) in Theorem 1. Also, $\forall\, X \subseteq S,\ Y \subseteq T$,

$$\begin{aligned} e(X, Y) \geqslant |N(X) \cap Y| &= |N(X)| + |Y| - |N(X) \cup Y| \\ &\geqslant |X| + |Y| - |T|, \end{aligned}$$

which is (ii). The result follows from Theorem 1. //

Corollary 1.2. G has a 1-factor if and only if the following three conditions hold:

(i) $|N(X)| \geqslant |X| \quad \forall\, X \subseteq S$,

(ii) $|N(Y)| \geqslant |Y| \quad \forall\, Y \subseteq T$,

(iii) $|S| = |T|$.

Moreover, any two of these imply the third.

Proof. (i) & (ii) $\iff |N(X)| \geqslant |X| \quad \forall\, X \subseteq V(G)$
$\iff G$ has a 1-factor by Corollary 1.1,
$\implies$ (iii).

To see that (i) & (iii) $\implies$ (ii), let $Y \subseteq T$, and define $X := S \setminus N(Y)$. Then $N(X) \cap Y = \emptyset$ and

$$\begin{aligned} |N(Y)| = |S| - |X| &= |T| - |X| && \text{by (iii)} \\ &\geqslant |T| - |N(X)| && \text{by (i)} \\ &\geqslant |Y|. \end{aligned}$$

Similarly (ii) & (iii) $\implies$ (i). //

If $S \subseteq V(G)$ (where now G is not necessarily bipartite), let $o(S) = o_G(S)$ denote the number of *odd components* (components with an odd number of vertices) in $G - S$.

Theorem 2. (W. T. Tutte, 1947.) G has a 1-factor if and only if $o(S) \leqslant |S|\ \forall\, S \subseteq V(G)$.

Proof. (I. Anderson, 1971.) 'Only if' holds because in any 1-factor of G, at least one vertex of S is matched with each odd component of $G - S$. We prove 'if' by induction on $|V(G)|$. Assume $o(S) \leqslant |S|$ $\forall S \subseteq V(G)$. Taking $S = \emptyset$ we see that $|V(G)|$ is even. Hence $o(S) \equiv |S|$ (mod 2), and

$$\text{if } o(S) \neq |S| \text{ then } o(S) \leqslant |S| - 2. \qquad (*)$$

If $|S| = 1$ then clearly $o(S) = |S|$. Choose S *maximal* such that $o(S) = |S|$. Then $G - S$ has $|S|$ odd components $C_1, \ldots, C_{|S|}$ — and no even components, otherwise transferring a vertex from one of them to S would increase $|S|$ and $o(S)$ by one and contradict the maximality of S. Let $T = \{1, 2, \ldots, o(S)\}$, and form a bipartite graph H on $S \cup T$ by joining s to t iff $s \in N_G(C_t)$. For each $Y \subseteq T$,

$$|N_H(Y)| \geqslant o_G(N_H(Y)) \geqslant |Y|$$

since $G - N_H(Y)$ has at least the $|Y|$ odd components $C_t : t \in Y$. By Corollary 1.2, H has a 1-factor, and so we can take a vertex x_i

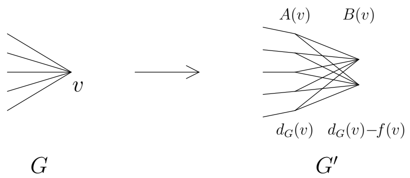
in each C_i and match it with a vertex in S. Let $U \subseteq V(C_i - x_i)$, for some i. Then

$$\begin{aligned} o_{C_i - x_i}(U) &= o_G(U \cup \{x_i\} \cup S) - (o_G(S) - 1) \\ &\leqslant |U \cup \{x_i\} \cup S| - 2 - (|S| - 1) \\ &\qquad\qquad \text{by the maximality of } S \text{ and } (*) \\ &= |U|. \end{aligned}$$

By the induction hypothesis each set $C_i - x_i$ has a 1-factor, and hence so does G. //

Theorem 3. (W. T. Tutte, 1952.) G has an f-factor if and only if, for every pair of disjoint subsets A, B of $V(G)$, the number of components C of $G - (A \cup B)$ for which $e(B, C) + \sum_{v\in C} f(v)$ is odd, is at most $\sum_{v\in A} f(v) + \sum_{v\in B} \left[d_{G-A}(v) - f(v)\right]$.

Proof. Both the existence of an f-factor, and the condition (with $A = \emptyset$, $B = \{v\}$) require $f(v) \leqslant d(v)\ \forall\, v \in V(G)$; so suppose this is true. We may assume $d(v) \neq 0$, $\forall v \in V(G)$. Form G' from G by carrying out the following construction at every vertex v:



Then G has an f-factor iff G' has a 1-factor. We must prove that the given condition is equivalent to $o_{G'}(S) \leqslant |S|$, $\forall\, S \subseteq V(G')$. Choose S minimal s.t. $|S| - o_{G'}(S)$ takes its minimum value.

Claim 1. $\forall\, v \in V(G)$, either $\emptyset \neq B(v) \subseteq S$ or $B(v) \cap S = \emptyset$: otherwise, removing a vertex w of $B(v)$ from S would reduce $|S|$

by 1 and change $o_{G'}(S)$ by at most one (since all neighbours of w in $G' - S$ are in the same component), thereby violating the choice of S.

Claim 2. If $B(v) \subseteq S$ then $A(v) \cap S = \emptyset$ (for the same reason, removing a vertex w of $A(v)$ from S).

Claim 3. $\forall\, v \in V(G)$, either $A(v) \subseteq S$ or $A(v) \cap S = \emptyset$. This follows from Claim 2 if $B(v) \subseteq S$, and so by Claim 1 we may assume that $B(v) \neq \emptyset$ and $B(v) \cap S = \emptyset$. Now a vertex $w \in A(v) \cap S$ has neighbours in at most two components of $G' - S$, and the result follows by the same reasoning as before.

Let $A := \{v \in V(G) : A(v) \subseteq S\}$
and $B := \{v \in V(G) : B(v) \subseteq S\}$.

Then A and B are disjoint by Claim 2, and, by Claims 1 and 3, $A(v) \cap S = \emptyset$ if $v \notin A$ and $B(v) \cap S = \emptyset$ if $v \notin B$. Each component C of $G - (A \cup B)$ corresponds to a component of $G' - S$ with $\sum_{v \in C} \big[2d_G(v) - f(v)\big] + e(B, C)$ vertices, which is odd iff $e(B,C) + \sum_{v \in C} f(v)$ is. The other odd components of $G' - S$ are all isolated vertices: $\sum_{v \in A} \big[d_G(v) - f(v)\big]$ of them corresponding to vertices in A, and $e(A,B) = \sum_{v \in B} d_A(v)$ of them corresponding to edges between A and B. Thus $o_{G'}(S) \leqslant |S|$ iff the number of components C of $G - (A \cup B)$ for which $e(B,C) + \sum_{v \in C} f(v)$ is odd is at most

$$|S| - \textstyle\sum_{v \in A} \big[d_G(v) - f(v)\big] - \sum_{v \in B} d_A(v). \qquad (1)$$

Since

$$|S| = \textstyle\sum_{v \in A} d_G(v) + \sum_{v \in B} \big[d_G(v) - f(v)\big],$$

(1) is equal to

$$\textstyle\sum_{v \in A} f(v) + \sum_{v \in B} \big[d_{G-A}(v) - f(v)\big],$$

as required.

It follows that if the condition given in the Theorem holds, then $o_{G'}(S) \leqslant |S|$, $\forall\, S \subseteq V(G')$. To prove the converse, let A and B be disjoint subsets of $V(G)$, and let

$$S := \bigcup\{A(v) : v \in A\} \cup \bigcup\{B(v) : v \in B\} \subseteq V(G').$$

Then the results of Claims 1–3 hold by definition, and the above argument shows that if $o_{G'}(S) \leqslant |S|$ then the condition in the Theorem holds for A and B. //

[In fact, the necessity of the condition in Theorem 3 is not difficult to see directly. Suppose G has an f-factor F. Then the number of ends of edges of F in B is $\sum_{v \in B} d_F(v) = \sum_{v \in B} f(v)$. Of the other ends of these edges, $e_F(A,B)$ are in A, and $\sum_{v \in B} d_{F-A}(v)$ are not in A. However, for each component of $G - (A \cup B)$ for which $e(B,C) + \sum_{v \in C} f(v)$ is odd, the number of edges between A and C that are in F, plus the number between B and C that are not in F, is odd. So if there are s such components, then $e_F(A,B) + \sum_{v \in B} d_{F-A}(v) \leqslant \sum_{v \in A} f(v) + \sum_{v \in B} d_{G-A}(v) - s$. Thus $\sum_{v \in B} f(v) \leqslant \sum_{v \in A} f(v) + \sum_{v \in B} d_{G-A}(v) - s$, which is the given condition.]

A multigraph G has the *odd cycle property* (*OCP*) if each two of its odd circuits either overlap (in at least one vertex) or are joined by an edge. Equivalently, if $S \subseteq V(G)$, then at most one component of $G - S$ contains an odd circuit.

Theorem 4. If G has the OCP then G has a 1-factor if and only if $|V(G)|$ is even and $|N(X)| \geqslant |X|\ \forall\, X \subseteq V(G)$ s.t. X is independent.

Proof. The conditions are clearly necessary. To prove they are sufficient, we use Theorem 2. Suppose $\exists\, S \subseteq V(G)$ s.t. $o(S) > |S|$. Since $|V(G)|$ is even, $o(S) \geqslant |S| + 2$. Since G has the OCP, at

most one component of $G - S$ contains an odd circuit. Let C_i $(i = 1, 2, \ldots, |S|+1)$ be bipartite odd components of $G - S$. Let X_i be the larger partite set of C_i, so that $|N_{C_i}(X_i)| \leqslant |X_i| - 1$, and let $X := \bigcup_{i=1}^{|S|+1} X_i$. Then X is independent and

$$\begin{aligned} |N(X)| &\leqslant |S| + \textstyle\sum_{i=1}^{|S|+1} |N_{C_i}(X_i)| \\ &\leqslant |S| + \textstyle\sum_{i=1}^{|S|+1} (|X_i| - 1) \\ &= |S| + |X| - |S| - 1 < |X|. \end{aligned}$$

This contradiction shows that $o(S) \leqslant |S|$, $\forall\, S \subseteq V(G)$, so that G has a 1-factor by Theorem 2. //

Theorem 5. (D. R. Fulkerson, A. J. Hoffman and M. H. McAndrew, 1965; Canad. J. Math. 17, 166–177.) If G has the OCP then G has an f-factor if and only if $\sum_{v \in V(G)} f(v)$ is even and

$$t(A,B) := \textstyle\sum_{v \in A} f(v) + \sum_{v \in B} \big[d_{G-A}(v) - f(v)\big] \geqslant 0$$

for every pair of disjoint subsets A, B of $V(G)$.

Proof. Let $s(A,B)$ denote the number of odd components C of $G - (A \cup B)$, where *odd* means now that $e(B,C) + \sum_{v \in C} f(v)$ is odd. By Tutte's f-factor theorem (Theorem 3),

$$\begin{aligned} G \text{ has an } f\text{-factor} &\iff s(A,B) \leqslant t(A,B) \quad \forall\, A, B \\ &\implies \textstyle\sum_{v \in V(G)} f(v) \text{ is even and} \\ &\qquad\qquad t(A,B) \geqslant 0 \quad \forall\, A, B. \end{aligned}$$

So suppose $\sum_{v \in V(G)} f(v)$ is even and $t(A,B) \geqslant 0\ \forall\, A, B$, and suppose $s(A,B) > t(A,B)$ for some A, B. Choose A, B such that $A \cup B$ is maximal with this property. Note that $s(A,B) + t(A,B)$ has the same parity as

$$r(A,B) := \sum_{\substack{C \text{ a cpt of} \\ G-(A \cup B)}} \big[e(B,C) + \textstyle\sum_{v \in C} f(v)\big] + t(A,B).$$

But

$$\textstyle\sum_{v \in B} d_{G-A}(v) = 2e(B,B) + e(B, V(G) - (A \cup B)),$$

and so $r(A,B)$ has the same parity as

$$2e(B, V(G) - (A \cup B)) + \textstyle\sum_{v \in V(G)} f(v),$$

which is even. Thus $s(A,B)$ has the same parity as $t(A,B)$, and $s(A,B) \geqslant t(A,B) + 2 \geqslant 2$. At most one odd component of $G - (A \cup B)$ contains an odd circuit, since G has the OCP; so let C be one that is bipartite, with bipartition (X, Y), say. The two quantities

$$r_X := e(B,X) - \textstyle\sum_{v \in X} f(v)$$

and

$$r_Y := e(B,Y) - \textstyle\sum_{v \in Y} f(v)$$

are not equal, since their sum has the same parity as $e(B,C) + \sum_{v \in C} f(v)$, which is odd. Suppose w.l.o.g. $r_X > r_Y$. Let $A' := A \cup X$, $B' := B \cup Y$. Then $s(A', B') = s(A,B) - 1$, and, since $N(Y) \subseteq X \cup A \cup B = A' \cup B$,

$$\begin{aligned} t(A', B') &= t(A,B) + \textstyle\sum_{v \in X} f(v) - \sum_{v \in Y} f(v) - e(B,X) + e(B,Y) \\ &= t(A,B) - r_X + r_Y \\ &< t(A,B). \end{aligned}$$

So $s(A', B') > t(A', B')$. This contradicts the maximality of $A \cup B$, so we conclude that $s(A,B) \leqslant t(A,B)\ \forall\, A, B$, whence G has an f-factor by Theorem 3. //

Exercises. Theorem 5 $\Longrightarrow$ Theorem 1,
Theorem 3 $\Longrightarrow$ Theorem 1 (directly).

Problem. Is there any other class of multigraphs, other than bipartite and those with the OCP, for which there is a n.s.c. for the existence of an f-factor that is different from Theorem 3?

2. Sufficient conditions. I. Anderson (1971) proved that if $|V(G)|$ is even and $|N(X)| \geqslant \frac{4}{3}|X| \; \forall\, X \subseteq V(G)$ s.t. $|X| \leqslant \frac{3}{4}|V(G)|$, then G has a 1-factor. D. R. Woodall (1973) defined the *binding number* of G to be

$$\begin{aligned} \operatorname{bind}(G) &:= \min\left\{\frac{|N(X)|}{|X|} : \emptyset \subsetneqq X \subseteq V(G) \text{ and } N(X) \neq V(G)\right\} \\ &= \max\{c : |N(X)| \geqslant c|X| \; \forall\, X \subseteq V(G) \text{ s.t.} \\ &\qquad\qquad N(X) \neq V(G)\}. \end{aligned}$$

So Anderson's result says that if $|V(G)|$ is even and $\operatorname{bind}(G) \geqslant \frac{4}{3}$ then G has a 1-factor. The figure $\frac{4}{3}$ here is best possible in view of the graph $(t+2)K_3 + tK_1$ (where $+$ denotes 'join'), which has no 1-factor and has binding number

$$\frac{3(t+1)+t}{3(t+1)} = \frac{4t+3}{3t+3} \to \frac{4}{3}- \quad \text{as } t \to \infty.$$

However, the result can be improved in other ways.

Theorem 6. (*Fundamental Lemma.*) Let G be a multigraph and a, b, c, d real numbers such that $a, b \geqslant 0$. Then

$$a|N(X)| \geqslant b|X| + c|V(G)| + d \qquad \text{whenever } \emptyset \subsetneqq X \subseteq V(G) \tag{2}$$

if and only if

$$b|N(X)| \geqslant a|X| + (c+b-a)|V(G)| + d \qquad \text{whenever } X \subseteq V(G),\ N(X) \neq V(G). \tag{3}$$

Proof. Suppose (2) holds and let $Y \subseteq V(G)$, $N(Y) \neq V(G)$. Put $X := V(G) \setminus N(Y)$, so that $X \neq \emptyset$ and $N(X) \subseteq V(G) \setminus Y$. Then

$$\begin{aligned} b|N(Y)| &= b|V(G)| - b|X| \\ &\geqslant b|V(G)| - a|N(X)| + c|V(G)| + d \qquad \text{by (2)} \\ &\geqslant b|V(G)| - a|V(G)| + a|Y| + c|V(G)| + d \\ &= a|Y| + (c+b-a)|V(G)| + d. \end{aligned}$$

Thus (2) implies (3). The converse is proved similarly. //

G is *sesquiconnected* if G is connected and, $\forall\, v \in V(G)$, $G - v$ has at most two components. Let $\delta(G) := \min\{|N(v)| : v \in V(G)\}$. (If G is simple, this is the minimum degree.)

Theorem 7. If G is sesquiconnected, $n = |V(G)|$ is even, $\delta(G) \geqslant \frac{1}{4}(n+3)$ and

$$|N(X)| \geqslant \tfrac{1}{4}(2|X| + n - 5) \tag{4}$$

for every nonempty *independent* subset X of $V(G)$, then G has a 1-factor.

Proof. Suppose not. Then $\exists\, S \subseteq V(G)$ s.t. $o(S) \geqslant |S| + 2$. Since $\delta(G) \geqslant \frac{1}{4}(n+3)$, each of the odd components of $G - S$ has at least $\frac{1}{4}(n+3) + 1 - |S|$ vertices, and so

$$n \geqslant |S| + \big(|S|+2\big)\big(\tfrac{1}{4}(n+3) + 1 - |S|\big),$$

whence

$$|S|^2 - \tfrac{1}{4}(n+3)|S| + \tfrac{1}{2}(n-7) \geqslant 0,$$

i.e.,

$$\big(|S| - 2\big)\big(|S| - \tfrac{1}{4}(n-5)\big) \geqslant 1.$$

This is impossible if $2 \leqslant |S| \leqslant \frac{1}{4}(n-5)$. Since $|S| \geqslant 2$ by the sesquiconnectedness of G, $|S| > \frac{1}{4}(n-5)$. Let x of the odd components consist of a single vertex each, and let X be the set of these. If $x = 0$ then $n \geqslant |S| + 3(|S|+2)$ and $|S| \leqslant \frac{1}{4}(n-6)$, which is impossible. So $X \neq \emptyset$ and, by (4),

$$|S| \geqslant |N(X)| \geqslant \tfrac{1}{4}(2x + n - 5). \tag{5}$$

Also,

$$n \geqslant |S| + x + 3(|S| + 2 - x). \tag{6}$$

Adding 4 times (5) to (6) gives $0 \geqslant 6 - 5 = 1$, a contradiction. //

Corollary 7.1. If $n = |V(G)|$ is even and

$$|N(X)| \geqslant \tfrac{1}{4}(2|X| + n + 1) \qquad \text{whenever } \emptyset \subsetneqq X \subseteq V(G) \tag{7}$$

then G has a 1-factor.

Proof. Clearly (7) implies (4), and also implies $\delta(G) \geqslant \frac{1}{4}(n+3)$ (take $|X| = 1$). If G is disconnected, let X be the vertex-set of a smallest component, for which $|N(X)| \leqslant |X| \leqslant \frac{1}{2}n$. Then (7) gives

$$|X| \geqslant \tfrac{1}{4}(2|X| + n + 1) \Longrightarrow \tfrac{1}{2}|X| \geqslant \tfrac{1}{4}(n+1) \Longrightarrow |X| \geqslant \tfrac{1}{2}(n+1),$$

a contradiction. If $G - v$ has more than two components, let X be the vertex-set of a smallest one, for which $|N(X)| \leqslant |X| + 1$ and $|X| \leqslant \frac{1}{3}(n-1)$. Then (7) gives

$$|X| + 1 \geqslant \tfrac{1}{4}(2|X| + n + 1) \Longrightarrow |X| \geqslant \tfrac{1}{2}(n-3) > \tfrac{1}{3}(n-1)$$

if $n > 7$. But if $n \leqslant 7$ then $n \leqslant 6$ (because n is even), so $|X| \leqslant \frac{5}{3}$, i.e., $|X| = 1$. But then $|N(X)| = |X|$, and we have already seen that it is impossible that $|N(X)| \leqslant |X| \leqslant \frac{1}{2}n$. So (7) implies that G is sesquiconnected, and so G has a 1-factor by Theorem 7. //

Corollary 7.2. If $n = |V(G)|$ is even then G has a 1-factor if:

(a) (I. Anderson, 1971) $\operatorname{bind}(G) \geqslant \frac{4}{3}$;

(b) (I. Anderson, 1972) $|N(X)| \geqslant \frac{1}{2}(4|X| - n + 1)$ whenever $X \subseteq V(G)$, $N(X) \neq V(G)$.

Proof. (a) $\Longleftrightarrow \frac{3}{4}|N(X)| \geqslant |X|$ whenever $X \subseteq V(G)$, $N(X) \neq V(G)$
$\Longleftrightarrow |N(X)| \geqslant \frac{3}{4}|X| + \frac{1}{4}n$ whenever $\emptyset \subsetneqq X \subseteq V(G)$ by Thm 6
$\Longrightarrow |N(X)| \geqslant \frac{1}{2}|X| + \frac{1}{4}n + \frac{1}{4}$ whenever $\emptyset \subsetneqq X \subseteq V(G) \Longleftrightarrow$ (7)
$\Longleftrightarrow \frac{1}{2}|N(X)| \geqslant |X| - \frac{1}{4}n + \frac{1}{4}$ whenever $X \subseteq V(G)$, $N(X) \neq V(G)$ by Thm 6

$\Longleftrightarrow$ (b).
The result follows from Corollary 7.1. //

There could be 12 theorems analogous to Theorem 7: for 1-factors, k-factors, $[a, b]$-factors and f-factors, and in arbitrary graphs, bipartite graphs and graphs with the OCP. The 1-factor theorems are given above, A. M. Robertshaw did k-factors in bipartite graphs, and T. R. Poole did $[a, b]$-factors in bipartite and arbitrary graphs. For k-factors in arbitrary graphs we have the following.

Theorem 8. (Woodall, 1990; Egawa and Enomoto, 1989.) Let G be a graph with n vertices, and let $k \geqslant 2$ be an integer. If k is odd, suppose n is even and G is connected. Suppose that

$$|N(X)| \geqslant \frac{1}{2k-1}\big(|X| + (k-1)n - 1\big)$$

for every nonempty independent subset X of $V(G)$, and

$$\delta(G) \geqslant \frac{(k-1)(n+2)}{2k-1}.$$

Suppose further that, if $n \leqslant 4k + 1 - 4\sqrt{k+2}$, then

$$\delta(G) > n + 2k - 2\sqrt{kn+2}.$$

Then G has a k-factor. //

Theorem 9. (C. Chen, 1995; DM 146, 303–306.) If $n = |V(G)|$ is even, $l \geqslant 1$ and

$$\operatorname{bind}(G) = b > \max\{l, \tfrac{7l+13}{12}\} = \begin{cases} \frac{5}{3} & \text{if } l = 1, \\ \frac{9}{4} & \text{if } l = 2, \\ l & \text{if } l \geqslant 3, \end{cases}$$

then every matching of l edges in G can be extended to a 1-factor.

Proof. Suppose not. Let L be the vertex-set of a matching that is not extendable, so that $|L| = 2l$. Then $\exists\, T \subseteq V(G-L)$ s.t. $o_{G-L}(T) \geqslant |T| + 2$; that is, writing $S := T \cup L$,

$$o_G(S) \geqslant |S| - 2l + 2. \tag{8}$$

Let $i(G-S)$ denote the number of isolated vertices in $G-S$. There are two cases.

Case 1: $i(G-S) = 0$. Let X be the vertex-set of any $|S| - 2l + 1$ odd components of $G-S$. Since $N(X) \neq V(G)$,

$$|X| + |S| \geqslant |N(X)| \geqslant b|X|,$$

and so $|X| \leqslant \frac{|S|}{b-1}$. But $|X| \geqslant 3(|S| - 2l + 1)$, and so

$$3(|S| - 2l + 1) \leqslant \tfrac{|S|}{b-1}. \tag{9}$$

Since $3(b-1) \geqslant 3(\frac{5}{3} - 1) > 1$ and $|S| \geqslant 2l$, (9) remains true with $|S|$ replaced by $2l$. Hence $3(b-1) \leqslant 2l$, and so $b \leqslant \frac{2}{3}l + 1$, a contradiction unless $l = 2$. If $l = 2$ and some odd component of $G-S$ has at least 5 vertices,, then the LHS of (9) is increased by 2 and we get $5(b-1) \leqslant 2l = 4$ or $b \leqslant \frac{9}{5}$, a contradiction. So we may assume that every odd component of $G-S$ has exactly 3 vertices. Add one vertex from another odd component to X, so that

$$|X| + |S| + 1 \geqslant |N(X)| \geqslant b|X|$$

and

$$3(|S| - 2l + 1) + 1 \leqslant |X| \leqslant \frac{|S| + 1}{b-1}.$$

This gives $4(b-1) \leqslant 2l + 1 = 5$, and so $b \leqslant \frac{9}{4}$, a contradiction.

Case 2: $i(G-S) > 0$. Let $X := V(G) - S$. Then

$$n - i(G-S) \geqslant |N(X)| \geqslant b|X| = bn - b|S|,$$

and so

$$b|S| - i(G-S) \geqslant (b-1)n.$$

From this and (8),

$$\begin{aligned} o_G(S) - i(G-S) &\geqslant |S| - 2l + 2 + (b-1)n - b|S| \\ &= (b-1)(n - |S|) - 2l + 2. \end{aligned}$$

But

$$\begin{aligned} n - |S| &\geqslant 3[o_G(S) - i(G-S)] + i(G-S) \\ &\geqslant (3b-3)(n-|S|) - 6l + 6 + i(G-S). \end{aligned}$$

Rearranging,

$$(3b-4)(n - |S|) \leqslant 6l - 6 - i(G-S). \tag{10}$$

Now, $n - |S| \geqslant o_G(S) \geqslant 2$, by (8), and so if $i(G-S) \geqslant 2$ then (10) gives $6b - 8 \leqslant 6l - 8$, whence $b \leqslant l$, a contradiction. But if $i(G-S) = 1$ then $n - |S| \geqslant 4$ (since the second odd component of $G-S$ has at least 3 vertices), and so $12b - 16 \leqslant 6l - 7$, giving the contradiction $b \leqslant \frac{6l+9}{12} < \frac{7l+13}{12}$. In either case, the theorem is proved. //

Robertshaw and Woodall replaced the binding-number condition in Theorem 9 by the correct (best possible) condition on $|N(X)|$ for X independent as in Theorem 7. Poole extended this to k-factors in bipartite graphs, and Philpotts and Woodall extended it to k-factors in arbitrary graphs. Nothing is known about f-factors or graphs with the OCP.