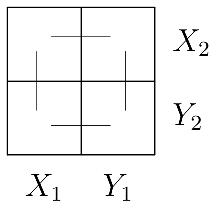
Chapter 1 Cycle-Coboundary Duality

Let G be a multigraph (loops permitted) with n vertices, m edges and c components. A *cycle* in G is a set of edges forming a subgraph in which every vertex has even degree. A *coboundary* in G is a set of edges of the form

$$\langle X, Y\rangle := \{xy \in E(G) : x \in X, y \in Y\}$$

for some $X \subseteq V(G)$, $Y = V(G) \setminus X$. A *cutset* in G is a set of edges whose removal increases the number of components, and a *cocircuit* is a minimal cutset. The *Boolean sum* (or *symmetric difference*) of sets A and B is $A +_2 B := (A \cup B) \setminus (A \cap B)$. Then $A_1 +_2 \ldots +_2 A_k$ is the set of elements that are contained in an odd number of $A_1, \ldots, A_k$. Let U be the m-dimensional vector space over GF(2) whose elements are the subsets of $E(G)$, with $+_2$ as addition, and scalar multiplication given by $0S := \emptyset$, $1S := S$. Let W (resp. W^*) be the set of cycles (resp. coboundaries).

Theorem 1.1. (a) If C_1 and C_2 are cycles (resp. coboundaries), then $C_1 +_2 C_2$ is a cycle (resp. coboundary). (So W and W^* are subspaces of U.)



Proof. This is true for cycles, since the Boolean sum of two even sets is even. And if $\langle X_1, Y_1\rangle$ and $\langle X_2, Y_2\rangle$ are coboundaries, then

$$\langle X_1, Y_1\rangle +_2 \langle X_2, Y_2\rangle = \langle (X_1 \cap X_2) \cup (Y_1 \cap Y_2),\ (X_1 \cap Y_2) \cup (X_2 \cap Y_1)\rangle. \ //$$

(b) The minimal nonempty cycles (coboundaries) are precisely the circuits (cocircuits). The cycles (coboundaries) are precisely the edge-disjoint unions of circuits (cocircuits), and also are precisely the Boolean sums of circuits (cocircuits).

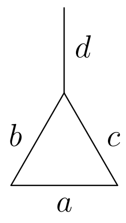
Proof. For cycles this all follows from (a) and the theorem that a nonempty cycle is an edge-disjoint union of circuits (pre-Euler's theorem). For coboundaries, note that every nonempty coboundary is a cutset. And if C is a cutset, let X be the set of vertices of a component of $G \setminus C$ that is not a component of G and note that $\langle X, V(G) \setminus X \rangle \subseteq C$; thus every cutset contains a nonempty coboundary. It follows immediately that the minimal nonempty coboundaries are precisely the minimal cutsets, i.e., cocircuits. The other two parts follow inductively, since if C is a nonempty coboundary and $C' \subseteq C$ is a cocircuit then $C \setminus C' = C +_2 C'$ is a smaller coboundary than C (using (a)). //

(c) If C_1 and C_2 are distinct circuits (cocircuits) both containing an edge e, then there is a circuit (cocircuit) in $(C_1 \cup C_2) \setminus \{e\}$.

Proof. $C_1 +_2 C_2$ does not contain e and, by (a) and (b), it is an edge-disjoint union of circuits (cocircuits). //

(d) A set of edges of G forms a cycle (coboundary) if and only if it has even intersection with every coboundary (cycle), or, equivalently, with every cocircuit (circuit). (So W and W^* are orthogonal complements in U with respect to the dot product over the basis of singletons.)

Example.



Cycles: $\emptyset$, $\{a, b, c\}$.
Coboundaries: $\emptyset$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$, $\{d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

$$\begin{array}{ll} & \scriptstyle a\ \ b\ \ c\ \ d \\ W\colon & (0,0,0,0), \\ & (1,1,1,0). \end{array} \qquad \begin{array}{lll} W^*\colon & (0,0,0,0), & (0,0,0,1), \\ & (1,1,0,0), & (1,1,0,1), \\ & (1,0,1,0), & (1,0,1,1), \\ & (0,1,1,0), & (0,1,1,1). \end{array}$$

Proof. A circuit clearly 'crosses' a coboundary an even number of times. By (b), this proves 'only if' in both cases. Let $S \subseteq E(G)$. If S has even intersection with every coboundary, then it does so with every *vertex coboundary* $\langle \{v\},\ V(G) \setminus \{v\} \rangle$ and so is a cycle.

It remains to prove, by induction on $|S|$, that if S has even intersection with every cycle, then S is a coboundary $\langle X, Y \rangle$. This is obvious if $|S| = 0$ (take $X := \emptyset$); so suppose $|S| \geqslant 1$ and choose $s = ab \in S$. Let $G' := G \setminus \{s\}$ and $S' := S \setminus \{s\}$. S' has even intersection with every cycle in G' (since all cycles in G' are cycles in G), and so, by the induction hypothesis, S' is a coboundary $\langle X', Y' \rangle$ in G'. W.l.o.g. $a \in X'$. If $b \in Y'$ take $X := X'$ and $Y := Y'$. Otherwise, a and b must lie in different components of G', since if P were a path connecting them then $|P \cap S'|$ would be even and so $|(P \cup \{s\}) \cap S|$ would be odd, $\Rightarrow\Leftarrow$, since $P \cup \{s\}$ is a circuit; so we obtain X and Y from X' and Y' by swapping over the vertices in *one* of these components. //

Let S be a *skeleton* (maximal subforest) of G, and $\overline{S} := E(G) \setminus S$.

(e) S contains no nonempty cycle, and $\overline{S}$ contains no nonempty coboundary.

Proof. S is a maximal circuit-free subset of $E(G)$ and so contains no nonempty cycle. Also, S contains a spanning tree of each component of G; thus every nonempty coboundary contains an edge of S, and so is not contained in $\overline{S}$. //

(f) If $e \in S$ and $f \in \overline{S}$, then $S \cup \{f\}$ ($\overline{S} \cup \{e\}$) contains a unique circuit C_f (cocircuit D_e), and $e \in C_f$ iff $f \in D_e$. Hence the skeletons (complements of skeletons) are precisely the maximal circuit-free (cocircuit-free) subsets of $E(G)$.

Proof. The existence of C_f follows from the definition of *skeleton*. For D_e, note that the removal of e from S disconnects some component of S into two, with vertex-sets X, Y, say. Let

$$D_e := \{xy \in E(G) : x \in X,\ y \in Y\}.$$

Then $e \in D_e \subseteq \overline{S} \cup \{e\}$. Also $D_e = \langle X,\ V(G) \setminus X \rangle$, which is a coboundary and so contains a cocircuit D'_e by (b). Let D''_e be any cocircuit contained in $\overline{S} \cup \{e\}$. Then D''_e is a nonempty coboundary by (b) and so contains e by (e), so that $D_e +_2 D''_e$ is a coboundary (by (a)) contained in $\overline{S}$, hence empty (by (e)), whence $D_e = D''_e = D'_e$. The uniqueness of C_f is proved similarly. And $f \in D_e$ iff f joins vertices in X and Y, which is precisely when $e \in C_f$.

The final statement follows because every circuit-free (cocircuit-free) subset of $E(G)$ is contained in a skeleton (complement of a skeleton), since the removal of a cocircuit-free subset from $E(G)$ does not increase the number of components. //

(g) $\{C_f : f \in \overline{S}\}$, the *system of fundamental circuits* associated with S, is a basis for W, which therefore has dimension $|E(G)| - |S| = m - n + c$, the *circuit rank* or *cycle rank* of G. $\{D_e : e \in S\}$, the *system of fundamental cutsets* (or *cocircuits*) associated with S, is a basis for W^*, which therefore has dimension $|S| = n - c$, the *cutset rank* of G.

Proof. Linear independence holds because each C_f contains an edge (f) in none other. If C is any cycle, let

$$C' := \textstyle\sum_2 \{C_f : f \in \overline{S} \cap C\}.$$

Then $C +_2 C' \subseteq S$ and is a cycle, hence $C = C'$ by (e). Thus $\{C_f : f \in \overline{S}\}$ is a basis for W. The rest is similar or straightforward. //

Exercises.

1. If $e \in E(G)$, let $G - e$ and G/e denote the results of deleting and contracting e. Let $C \subseteq E(G) \setminus \{e\}$. Then
   (a) C is a cycle in $G - e \iff C$ is a cycle in G.
   (b) C is a cycle in $G/e \iff C$ or $C \cup \{e\}$ is a cycle in G.
   (c) C is a coboundary in $G - e \iff C$ or $C \cup \{e\}$ is a coboundary in G.
   (d) C is a coboundary in $G/e \iff C$ is a coboundary in G.
2. Two graphs G and G^* are *abstract duals* or *combinatorial duals* if there is a bijection from $E(G)$ to $E(G^*)$ under which the cycles (coboundaries) of G correspond to the coboundaries (cycles) of G^*. If this holds, and e, e^* are corresponding edges of G, G^*, then $G - e$, G^*/e^* are abstract duals, and so are G/e, $G^* - e^*$.
3. Let $C \subseteq E(G)$. C is a coboundary iff C is a Boolean sum of vertex coboundaries. If G is a plane multigraph, then C is a cycle iff C is a Boolean sum of face cycles.
4. Geometric duals (of planar multigraphs) are abstract duals. (The converse is 'almost' true — it is true for loopless 2-connected multigraphs.)
5. A multigraph is planar iff it has an abstract dual.