Colourfully Panconnected Subgraphs

A set $X \subseteq V(G)$ is *distance-k-connected* in G if, $\forall\, u, w \in X$, there is a sequence $u = x_0, x_1, \ldots, x_l = w$ of vertices of X such that the distance $d_G(x_{i-1}, x_i) \leqslant k$ for each i $(1 \leqslant i \leqslant l)$.

Theorem 1. (J. Ouyang (unpublished manuscript, 1994), J. Griggs and E. Czabarka.) If G is a connected k-chromatic graph, then G has a (proper vertex) k-colouring such that each colour class is distance-k-connected in G.

This was conjectured by Chen, Schelp and Shreve (DM 170 (1997) 231–236), who note that 'distance-k-connected' is best possible—consider two copies of K_k connected by a long path.

If $v \in V(G)$, a k-colouring of G is:
variegated at v if the k colours can be ordered as $c_0, c_1, \ldots, c_{k-1}$ so that, $\forall\, i$, colour c_i occurs within distance i of v (so v has colour c_0);
variegated if it is variegated at every vertex;
panconnected if, for each i $(1 \leqslant i \leqslant k)$, the union of any i colour classes is distance-$(k{+}1{-}i)$-connected in G.

Ouyang's proof of Theorem 1 can be modified to prove:

Theorem 2. (O. V. Borodin and D. R. Woodall, 1996.) Let G be a connected k-colourable graph with at least k vertices. Then G has a variegated panconnected k-colouring.

A subgraph H of G is *k-colourfully panconnected* (*k-cp*) in G if G has a k-colouring that induces a variegated panconnected k-colouring of H; clearly then H is connected and $|V(H)| \geqslant k$. If G is a connected k-colourable graph with $n \geqslant k$ vertices, let

$$s_k(G) := \min\{|E(H)| : H \text{ is a spanning } k\text{-cp subgraph of } G\};$$

this makes sense because, by Theorem 2, G is a spanning k-cp subgraph of itself. If G is bipartite then clearly $s_2(G) = n - 1$: any spanning tree will do for H. So suppose $k \geqslant 3$.

Conjecture. (Borodin and Woodall, 1996.) If G is a connected k-colourable graph with $n \geqslant k \geqslant 4$ vertices, then $s_k(G) \leqslant n$; that is, G has an at-most-unicyclic spanning k-cp subgraph. [Moreover, if G has an edge that is contained in no circuit with length $\equiv 1 \pmod{k}$, then $s_k(G) = n - 1$.]

Theorem 3. Assume $n \geqslant k \geqslant 3$.

(a) $s_k(C_n) = \begin{cases} n & \text{if } n \equiv 1 \pmod{k}, \\ n-1 & \text{otherwise.} \end{cases}$

(b) $s_k(K_{r,s}) = \begin{cases} n + \delta - 2 & \text{if } k = 3, \\ n-1 & \text{if } k \geqslant 4, \end{cases}$
where $n = r + s$, $\delta = \min\{r, s\}$.

Proof. (a) Clearly $s_k(C_n) = n - 1$ or n. Let $P : v_0 v_1 \ldots v_{n-1}$ be a spanning tree (path) of C_n. It is easy to see that $c : V(P) \to \{0, 1, \ldots, k-1\}$ is a variegated panconnected k-colouring of P if and only if (after possibly relabelling the colours) $c(v_i) \equiv i \pmod{k}$, $\forall\, i$. This gives a proper k-colouring of C_n iff $n \not\equiv 1 \pmod{k}$. If $n \equiv 1 \pmod{k}$, recolour v_{n-1} with colour 1 to give a variegated panconnected k-colouring of C_n itself.

(b) Let $G = K_{r,s}$. If $\delta = 1$, then $s_k(G) = n - 1$ for all k. (Any colouring using all k colours will do.) If $\delta \geqslant 2$ and $k \geqslant 4$, then colourings and spanning trees as in Fig. 1(a) will work. So suppose $\delta \geqslant 2$ and $k = 3$. It follows from Theorem 5 or Fig. 1(b) that $s_3(G) \leqslant n + \delta - 2$. We must prove that $s_3(G) \geqslant n + \delta - 2$.

Let G have partite sets X, Y. Let H be a spanning 3-cp subgraph of G, and let $c : V(G) \to \{0, 1, 2\}$ be a 3-colouring that induces a variegated panconnected 3-colouring of H. Let X_i

be the set of vertices with colour i ($i = 0, 1, 2$). W.l.o.g. $X = X_0$, $Y = X_1 \cup X_2$. Each vertex v of X is adjacent in H to vertices in X_1 and X_2 since $c|_H$ is variegated at v. So, for each such v, choose edges vv_1, vv_2 of H with $v_1 \in X_1$, $v_2 \in X_2$ and let H' be the subgraph of H comprising the $2|X|$ edges so chosen. Note that, in H', $d(u, w) \geqslant 4$ if $u, w \in X_1$ or $u, w \in X_2$. Since X_1 and X_2 are distance-3-connected in H, $E(H) \setminus E(H')$ must contain at least $|X_i| - 1$ edges incident with X_i ($i = 1, 2$). Hence

$$|E(H)| \geqslant 2|X_0| + (|X_1| - 1) + (|X_2| - 1) = n + |X_0| - 2 \geqslant n + \delta - 2.$$

It follows that $s_3(G) \geqslant n + \delta - 2$, as required. //

Theorem 4. Let v_0v_1 be an edge in a connected k-colourable graph G. Then there is a k-colouring $c : V(G) \to \{0, 1, \ldots, k-1\}$, and a spanning tree T of G containing edge v_0v_1, such that $c(v) \equiv d_T(v_0, v) \pmod{k}$, $\forall\, v \in V(G)$.

Proof. Choose a k-colouring c of G and permute colours if necessary so that $c(v_0) = 0$, $c(v_1) = 1$. Set $T = \{v_0, v_0v_1, v_1\}$.

Construction 1. (Growing the tree.) While $\exists\, uw \in E(G)$ s.t. $u \in V(T)$, $w \in V(G) \setminus V(T)$ and $c(w) \equiv c(u) + 1 \pmod{k}$, add w and uw to T. If now (when $\nexists$ such an edge uw) $V(T) = V(G)$, then T is the required spanning tree, so stop; otherwise, invoke Construction 2.

Construction 2. (Changing the colouring.) Choose $uw \in E(G)$ s.t. $u \in V(T)$, $w \in V(G) \setminus V(T)$ and $c(w) \equiv c(u) + r \pmod{k}$, where $r \geqslant 2$ is as small as possible. Set $T' = \{w\}$, and while $\exists\, w' \in V(T')$, $u' \in V(G) \setminus V(T')$ s.t. $u'w' \in E(G)$ and $c(u') \equiv c(w') - 1 \pmod{k}$, add u' and $u'w'$ to T'. Note that $T' \cap T = \emptyset$, since if $c(u') \equiv c(w') - 1$ then $c(w') \equiv c(u') + 1$ and so if $u' \in T$ then $w' \in T$, $\Rightarrow\Leftarrow$. Now reduce $c(v)$ by 1 (mod k), $\forall\, v \in V(T')$. The new colouring c is still

proper, by the definition of T', and now $c(w) \equiv c(u) + r - 1 \pmod{k}$. If $r - 1 > 1$, repeat Construction 2 a further $r - 2$ times. Eventually $c(w) \equiv c(u) + 1 \pmod{k}$, and w and uw can be added to T. So return to Construction 1, and iterate until $V(T) = V(G)$. //

Corollary 4.1. Let G be a connected k-chromatic graph that contains a vertex v_0 that is adjacent to vertices of all other colours in *every* k-colouring of G. (E.g., $k = 3$ and G contains a triangle, or $k = 4$ and G contains a wheel with odd circumference.) Then $s_k(G) \leqslant n$.

Proof. Choose v_1 arbitrarily in $N(v_0)$, and form T and a k-colouring of G as in Theorem 4. Let e be any edge of G joining v_0 to a vertex of colour $k - 1$, and let $H := T \cup \{e\}$. Then H is a k-cp subgraph of G with n edges. //

A graph is r-*degenerate* if every subgraph of it contains a vertex with degree $\leqslant r$, or, equivalently, if its vertices can be ordered so that each is adjacent to at most r *earlier* vertices.

Theorem 5. Let G be a connected k-colourable graph with $n \geqslant k$ vertices, where $k = 3$ or 4. Then G has a 2-degenerate spanning k-cp subgraph with at most $n + \delta - 2$ edges; hence $s_k(G) \leqslant n + \delta - 2$. Moreover, if G has an edge e that is contained in no circuit with length $\equiv 1 \pmod{k}$, then $s_k(G) = n - 1$.

Proof. Form T and a k-colouring of G as in Theorem 4, where v_0v_1 is the edge e if it exists and is otherwise arbitrary subject to $d(v_0) = \delta$.

If T has no vertex of colour 2, then G is a star and the result holds (cf. Theorem 3(b)). If $k = 4$ and T has no vertex of colour 3, then recolour an arbitrary T-neighbour of v_0 with colour 3 unless $d_T(v_0) = 1$, in which case recolour an arbitrary T-neighbour of v_1

(other than v_0) with colour 3. In either case, T is a k-cp subgraph of G and $s_k(G) = n-1$; and this also holds whenever $d_T(v_0) = 1$. So we may suppose that all colours do occur in T and that $d_T(v_0) \geqslant 2$.

Let the components of $T - v_0$ be $T_1, \ldots, T_r$, where $v_1 \in T_1$. Let X_i be the set of vertices with colour i. T may fail to be a k-cp subgraph for three reasons:

(i) if there is 'short branch' (T_5 or T_3 in Fig. 2) then the colouring of T is not variegated at vertices in it;

(ii) X_{k-1} may not be distance k-connected in T (although $X_{k-1} \cap T_i$ is, $\forall\, i$);

(iii) if $k = 4$ then $X_2 \cup X_3$ may not be distance-3-connected in T (although, again, $(X_2 \cup X_3) \cap T_i$ is, $\forall\, i$).

If v_0v_1 is contained in no circuit of G with length $\equiv 1 \pmod k$, then we can interchange colours 1 and $k-1$ throughout T_1 to obtain a proper k-colouring of G that is a variegated panconnected k-colouring of T. So $s_k(G) = n-1$.

The same conclusion will follow if we can interchange colours 1 and $k-1$ throughout any proper nonempty subset of $T_1, \ldots, T_r$; so suppose we can't. Then we can reorder $T_2, \ldots, T_r$ if necessary so that, for each i ($1 \leqslant i \leqslant r-1$), there is an edge e_i of G joining a vertex of colour 1 or $k-1$ in $T_1 \cup \ldots \cup T_i$ to a vertex of colour $k-1$ or 1 in T_{i+1} (Fig. 3). Adding these $r-1$ edges to T will create a 2-degenerate spanning k-cp subgraph H of G with $n-1+r-1 \leqslant n+\delta-2$ edges. //

Proof of Theorem 1. The result is obvious if $k \leqslant 2$, and it follows from Theorem 5 if $k = 3$ or 4. The following argument works for all $k \geqslant 4$.

Let c be a k-colouring of a connected k-chromatic graph G. Let v be a vertex adjacent to vertices of all other colours (which exists since G is k-chromatic), and let $H := \langle \{v\} \cup N(v) \rangle_G$. Let

$\partial(H) := V(H) \cap N(V(G) \setminus V(H))$. We say that c is *strongly variegated* at a vertex x if, for each i ($2 \leqslant i \leqslant k-2$), within distance i of x there are vertices with at least $i+1$ colours different from that of x. Then the induced colouring c_H of H has the following properties:

P1. each c_H-colour-class is distance-k-connected in H;

P2. c_H is variegated;

P3. c_H is strongly variegated at all but at most one vertex in $\partial(H)$.

(In fact, P3 holds with no such exceptional vertex.) We now modify c on vertices outside H, and add vertices and edges to H, while preserving P1–P3, until $\partial(H) = \emptyset$, at which point the result will be proved.

So suppose $\partial(H) \neq \emptyset$ and choose $x \in \partial(H)$ such that c_H is strongly variegated at all vertices in $\partial(H) \setminus \{x\}$. Let $u \in N(x) \setminus V(H)$. Choose a colour c_x whose closest occurrence to x in H is as far from x as possible (at distance $\leqslant k-1$). There are three cases.

Case 1: $c(u) = c_x$. Then add u and the edge ux to H. It is easy to see that P1–P3 still hold, with u being the exceptional vertex in P3 if there is one. (This is possible even if x is not exceptional.)

Case 2: u is adjacent to a vertex y of colour c_x in H. Add u and the edges ux, uy to H. Then P1–P3 hold, with u again being the exceptional vertex if there is one.

Case 3: neither of the previous cases arises. For each vertex of colour $c(u)$ or c_x in $V(G) \setminus (V(H) \cup \{u\})$ in turn, change its colour to be different from both $c(u)$ and c_x if possible. Let C be the Kempe chain in G with colours $c(u)$ and c_x containing u. If $C \cap H = \emptyset$ then interchange colours in C and we are back in Case 1. Otherwise, $\exists\, w \in V(G) \setminus (V(H) \cup \{u\})$ and $z \in N(w) \cap V(H)$ s.t. $w, z \in C$. Note that w is adjacent to vertices of all other colours in G, since otherwise its colour would have been changed at the start of Case 3. Add w and all its neighbours and neighbouring edges to H. Since the old c_H was strongly variegated at z, P1–P3 still hold, with x remaining the exceptional vertex if there is one. //