The Reconstruction Conjecture

1. Definitions. Unless otherwise stated, 'graph' means 'multigraph'. If G is a graph and $u \in V(G)$, let $G_u := G - u$, a *vertex-deleted subgraph* of G; $\{G_u : u \in V(G)\}$ is the *deck* $D(G)$ of G. Two graphs G, H are *hypomorphic*, written $G \sim H$, if $|V(G)| \geqslant 3$ and there is a bijection $\phi : V(G) \to V(H)$ such that $G - u \cong H - \phi(u)$ for each u in $V(G)$; that is, G and H have 'the same' deck of vertex-deleted subgraphs. The *reconstruction conjecture* (S. M. Ulam and P. J. Kelly, 1942) is that $G \sim H \Longrightarrow G \cong H$; that is, if $|V(G)| \geqslant 3$, then G is uniquely determined by its deck of vertex-deleted subgraphs, in which case G is called *reconstructible*. ($G = K_2$ and $H = \bar{K}_2$ are counterexamples for $|V(G)| = 2$.) Clearly if G is simple then G is reconstructible iff $\bar{G}$ is.

A graph property or parameter is *reconstructible* if its truth or value for every graph G is determined by $D(G)$; that is, if $G \sim H \Longrightarrow$ the truth-value of the property or the value of the parameter is the same for G and H. A class $\mathcal{C}$ of graphs is *recognizable* if its defining property is reconstructible; that is, if $(G \sim H$ and $G \in \mathcal{C}) \Longrightarrow H \in \mathcal{C}$. Clearly $\mathcal{C}$ is recognizable iff $\bar{\mathcal{C}}$ is. $\mathcal{C}$ is *weakly reconstructible* if $(G, H \in \mathcal{C}$ and $G \sim H) \Longrightarrow G \cong H$. $\mathcal{C}$ is *reconstructible* if every member of $\mathcal{C}$ is reconstructible, i.e., $\mathcal{C}$ is both recognizable and weakly reconstructible. Classes that are known to be reconstructible include outerplanar graphs and (therefore) forests, unicyclic graphs and cacti. But it is not known whether the class of planar graphs is even recognizable.

2. Basic results. Let G have $n \geqslant 3$ vertices and m edges.

Theorem 1. n, m and the connectivity of G are reconstructible.

Proof. $n = |V(G_u)| + 1, \forall\, G_u \in D(G)$.
$m = \frac{1}{n-2} \sum_{G_u \in D(G)} |E(G_u)|$.

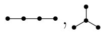
G is connected iff at least two of the graphs G_u are connected. (Exercise.) For $k \geqslant 2$, G is k-connected iff every graph G_u is $(k-1)$-connected. //

Theorem 2. (a) If G is disconnected then its set of components is reconstructible.
(b) If G is not 2-connected then its set of blocks is reconstructible.

Proof. We prove (b); (a) is similar. List all the blocks of all the graphs G_u. For each block B of G, the list contains $n - |V(B)|$ copies of B and one copy of every block of every vertex-deleted subgraph of B: call these the graphs *attributable* to B. Let B_1 be a largest graph in the list; B_1 must be a block of G. Delete from the list all graphs attributable to B_1. Let B_2 be a largest graph remaining in the list; B_2 must be a block of G. Delete from the list all graphs attributable to B_2. Repeat until the list is empty. //

Corollary 2.1. Disconnected graphs are reconstructible.

Proof. By Theorem 1, the class of disconnected graphs is recognizable. The result follows from Theorem 2 since a graph is uniquely determined by its set of components. //

Note that a graph is *not* determined by its set of blocks: .

Corollary 2.2. The number c of components, the circuit rank $m - n + c$ and the (cutset) rank $n - c$ are reconstructible, as are the orders of the components and the orders of the blocks. //

Theorem 3. The clique number $\omega(G)$, the vertex and edge independence numbers $\alpha(G)$ and $\alpha'(G)$, and the vertex and edge covering numbers $\beta(G)$ and $\beta'(G)$, are all reconstructible.

Proof. By a theorem of Gallai, $\alpha + \beta = \alpha' + \beta' = n$, and so it suffices to prove that ω, α and α' are reconstructible. Each of these is equal to the maximum value of the same parameter over all graphs G_u, unless ω or $\alpha = n$ or $\alpha' = \frac{1}{2}n$. But

$$\omega(G) = n \Longleftrightarrow G \text{ is complete} \Longleftrightarrow \text{every } G_u \text{ is complete},$$

$$\alpha(G) = n \Longleftrightarrow G \text{ is edgeless} \Longleftrightarrow \text{every } G_u \text{ is edgeless},$$

$$\begin{aligned}\alpha'(G) = \tfrac{1}{2}n &\Longleftrightarrow G \text{ has a perfect matching}\\ &\Longleftrightarrow \forall\, S \subseteq V(G),\ G - S \text{ has} \leqslant |S| \text{ odd components} \quad \text{(W. T. Tutte, 1947)}\\ &\Longleftrightarrow G \text{ has no odd component and } \forall\, G_u,\ \forall\, S \subseteq V(G_u),\\ &\qquad G_u - S \text{ has} \leqslant |S| + 1 \text{ odd components.}\end{aligned}$$

(Alternatively, use Theorem 6(b) below to prove directly that the number of perfect matchings is reconstructible.) //

Let $n(H, G)$ and $n_{\mathrm{I}}(H, G)$ denote the numbers of subgraphs and induced subgraphs of G isomorphic to H.

Theorem 4. *Kelly's Lemma* (P. J. Kelly, 1957). If $|V(H)| < n$ then $n(H, G)$ and $n_{\mathrm{I}}(H, G)$ are reconstructible.

Proof. $n_{(\mathrm{I})}(H, G) = \frac{1}{n - |V(H)|} \sum_{G_u \in D(G)} n_{(\mathrm{I})}(H, G_u)$. //

Corollary 4.1. The numbers of subgraphs and induced subgraphs of G with at most $n - 1$ vertices that are (a) contractible to H, and (b) homeomorphic to H, are reconstructible. //

Theorem 5. Suppose G is simple. For each G_u in $D(G)$, we can reconstruct (a) the degree $d_G(u)$, (b) $|E(\langle N_G(u) \rangle)|$, and (c) the degrees of the neighbours of u in G.

Proof. (a) $d_G(u) = |E(G)| - |E(G_u)| = n(K_2, G) - n(K_2, G_u)$.
(b) $|E(\langle N_G(u) \rangle)|$ is the number of triangles containing u in G, which is $n(K_3, G) - n(K_3, G_u)$.
(c) Let $d_1 \leqslant \ldots \leqslant d_{n-1}$ be the degrees of all vertices of G except u. Let $d'_1 \leqslant \ldots \leqslant d'_{n-1}$ be the degrees of all vertices of G_u. Then the subsequence $(d_i : d'_i < d_i)$ contains precisely the degrees of the neighbours of u in G. (This is because the degree of a vertex in G_u differs by *at most* 1 from its degree in G.) // $\begin{bmatrix} 2\,2\,3\,3\,4\,5 \\ 1\,2\,2\,3\,3\,5 \end{bmatrix}$

Corollary 5.1. Suppose G is simple. If, for all $v, w \in V(G)$, $d_G(v) - d_G(w) \neq 1$, then G is reconstructible. In particular, regular graphs and Eulerian graphs are reconstructible. More generally, if $\exists\, u \in V(G)$ s.t. $d_G(v) - d_G(w) \neq 1$ for all $v \in N_G(u)$ and $w \in V(G) \setminus N_G(u)$, then G is reconstructible.

3. Spanning subgraphs. (W. T. Tutte (1917–2002), 1979.) Let $\mathcal{S} = (X_1, \ldots, X_k)$ be a sequence of k graphs ($k \geqslant 1$). A *realization* of $\mathcal{S}$ in G is a sequence $H_1, \ldots, H_k$ of subgraphs of G, not necessarily distinct, such that $H_i \cong X_i$ for each i. The *carrier* of this realization is the graph $\bigcup_{i=1}^{k} H_i$. Let $\delta(\mathcal{S}, H)$ denote the number of realizations of $\mathcal{S}$ in H whose carrier is H. Let $n := |V(G)| \geqslant 3$ and let $\mathcal{H}$ denote the set of all nonisomorphic graphs with at most n vertices. We shall denote the sum

$$\sum_{H} \delta(\mathcal{S}, H) n(H, G)$$

by:
Σ_0 if the sum is over all $H \in \mathcal{H}$,
Σ_{vp} if the sum is over all $H \in \mathcal{H}$ s.t. $|V(H)| < n$,
Σ_{s} if the sum is over all $H \in \mathcal{H}$ s.t. $|V(H)| = n$,
Σ_{sd} if the sum is over all $H \in \mathcal{H}$ s.t. $|V(H)| = n$ & H is disconnected,
Σ_{sc} if the sum is over all $H \in \mathcal{H}$ s.t. $|V(H)| = n$ & H is connected.

Theorem 6. (a) If every graph in $\mathcal{S}$ has fewer vertices than G, then Σ_0, $\Sigma_{\rm vp}$, $\Sigma_{\rm s}$, $\Sigma_{\rm sd}$, and $\Sigma_{\rm sc}$ are all reconstructible.
(b) If K is a disconnected graph such that $|V(K)| = n$, then $n(K, G)$ is reconstructible. (This is 'Kelly's Lemma' for disconnected graphs.) [$\Longrightarrow \alpha'$ is reconstructible]

Proof. (a) Σ_0 is the total number of realizations of $\mathcal{S}$ in G, which is equal to $\prod_{i=1}^{k} n(X_i, G)$. This is reconstructible by Kelly's Lemma. So is $n(H, G)$ if $|V(H)| < n$, and so $\Sigma_{\rm vp}$ is reconstructible. Hence $\Sigma_{\rm s} = \Sigma_0 - \Sigma_{\rm vp}$ is reconstructible. It will follow from (b) that $\Sigma_{\rm sd}$ is reconstructible, and so $\Sigma_{\rm sc} = \Sigma_{\rm s} - \Sigma_{\rm sd}$ is reconstructible.

(b) Let $\mathcal{S} = (X_1, \ldots, X_k)$ where $X_1, \ldots, X_k$ are the components of K, and let $k_1, \ldots, k_r$ be the numbers of components in the different isomorphism classes, so that $k_1 + \ldots + k_r = k$. If H is a spanning subgraph of G then $|V(H)| = |V(K)| = \sum_{i=1}^{k} |V(X_i)|$, and

$$\begin{aligned} H \cong K &\Longleftrightarrow H \text{ is the carrier of a realization of } \mathcal{S} \text{ in } G \\ &\Longrightarrow \delta(\mathcal{S}, H) = \textstyle\prod_{i=1}^{r} k_i!, \end{aligned}$$

while

$$H \not\cong K \Longleftrightarrow \delta(\mathcal{S}, H) = 0.$$

Thus

$$n(K, G) = \frac{1}{\prod_{i=1}^{r} k_i!} \sum_{H}{}_{\rm s}\, \delta(\mathcal{S}, H) n(H, G),$$

which is reconstructible by (a). //

Theorem 7. If $X_1, \ldots, X_k$ ($k \geqslant 2$) are blocks, then the numbers N and $N_{\rm c}$ of subgraphs and connected subgraphs whose k blocks are isomorphic to $X_1, \ldots, X_k$ are reconstructible.

Proof. As in Theorem 6(b), let $\mathcal{S} := (X_1, \ldots, X_k)$ and let $k_1 \ldots, k_r$ be the numbers of blocks in the different isomorphism classes, so that

$k_1 + \ldots + k_r = k$. If X is a graph with k blocks isomorphic to $X_1, \ldots, X_k$ (and without isolated vertices) then

$$\textstyle\sum_{i=1}^{k} |V(X_i)| - k + 1 \leqslant |V(X)| \quad \Big(\leqslant \sum_{i=1}^{k} |V(X_i)|\Big),$$

with equality on the left iff X is connected. So if

$$\textstyle\sum_{i=1}^{k} |V(X_i)| - k + 1 > n$$

then $N = N_{\rm c} = 0$, and if

$$\textstyle\sum_{i=1}^{k} |V(X_i)| - k + 1 < n$$

then every such X *either* satisfies $|V(X)| < n$ *or* is disconnected, and so N and $N_{\rm c}$ are reconstructible by Kelly's Lemma and its extension (Theorems 4 and 6(b)). Thus it suffices to consider the case when

$$\textstyle\sum_{i=1}^{k} |V(X_i)| - k + 1 = n.$$

In this case, if X is a subgraph of G with k blocks isomorphic to $X_1, \ldots, X_k$, then X is connected and $|V(X)| = n$ and X is the carrier of a realization of $\mathcal{S}$ in G. So let H be a *connected* spanning subgraph of G. If H is the carrier of a realization $H_1, \ldots, H_k$ of $\mathcal{S}$ in G, then $|V(H)| = n = \sum_{i=1}^{k} |V(H_i)| - k + 1$, and it is not difficult to see that the blocks of H are $H_1, \ldots, H_k$ and hence are isomorphic to $X_1, \ldots, X_k$, so that $\delta(\mathcal{S}, H) = \prod_{i=1}^{r} k_i!$. If H is *not* the carrier of a realization of $\mathcal{S}$ in G, then $\delta(\mathcal{S}, H) = 0$. Thus

$$N = N_{\rm c} = \frac{1}{\prod_{i=1}^{r} k_i!} \sum_{H}{}_{\rm sc}\, \delta(\mathcal{S}, H) n(H, G),$$

which is reconstructible by Theorem 6(a). //

Let $n(q)$ be the number of spanning subgraphs of G with q edges,
$n_{\rm d}(q)$ be the number of these that are disconnected,
$n_{\rm c}(q)$ be the number that are connected,
$n_{\rm cs}(q)$ be the number that are connected and separable (not 2-connected),
$n_{\rm cn}(q)$ be the number that are connected and nonseparable (2-connected).

Theorem 8. $n(q)$, $n_{\rm d}(q)$, $n_{\rm c}(q)$, $n_{\rm cs}(q)$ and $n_{\rm cn}(q)$ are all reconstructible.

Proof. Clearly $n(q) = \binom{|E(G)|}{q}$. Also, $n_{\rm d}(q) = \sum_H n(H, G)$, where the sum is over all nonisomorphic disconnected graphs H such that $|V(H)| = n$ and $|E(H)| = q$, and each term $n(H, G)$ is reconstructible by Theorem 6(b). Hence $n_{\rm c}(q) = n(q) - n_{\rm d}(q)$ is reconstructible. Similarly, $n_{\rm cs}(q) = \sum_H n(H, G)$ where the sum is over all nonisomorphic connected graphs H with at least two blocks such that $|V(H)| = n$ and $|E(H)| = q$, and this sum is reconstructible by Theorem 7. Hence $n_{\rm cn}(q) = n_{\rm c}(q) - n_{\rm cs}(q)$ is reconstructible. //

Corollary 8.1. The number of hamiltonian circuits of G is reconstructible.

Proof. This is just $n_{\rm cn}(n)$. //

Corollary 8.2. The class of hamiltonian graphs is recognizable.

Corollary 8.3. The girth and circumference of G are reconstructible.

Proof. By Kelly's Lemma and Corollary 8.1, $n(C_k, G)$ is reconstructible for each k. //

Theorem 9. The number of hamiltonian paths of G is reconstructible.

Proof. For $t \geqslant 0$, let $G(t)$ denote the multigraph formed from G by adding t parallel edges between each pair of vertices of G (so that $G(0) = G$). Let $h(t)$ denote the number of hamiltonian circuits in $G(t)$, and let h_s denote the number of hamiltonian circuits in $G(1)$ that have exactly s edges not in G (where $G \subset G(1)$). Then (clearly)

$$h(t) = \textstyle\sum_{s=0}^{n} h_s t^s.$$

But, for each $u \in V(G)$, $G(t) - u \cong G_u(t)$, which is reconstructible from G_u. By Corollary 8.1, $h(t)$ is reconstructible for each t. The values of $h(t)$ for any $n + 1$ distinct values of t determine the coefficients h_s, which are therefore reconstructible. The number of hamiltonian paths of G is simply h_1. //

Theorem 10. The Tutte polynomial of G is reconstructible.

Proof. Let $N(k, q)$ denote the number of spanning subgraphs H of G with k components and q edges. Then

$$T(G, x, y) = \textstyle\sum_{k,q} N(k, q)(x - 1)^{k-c}(y - 1)^{q-n+k},$$

where c is the number of components of G, since

$$\gamma_G(H) = |E(H)| - r_G(H) = |E(H)| - n + c_G(H) = q - n + k.$$

Thus it suffices to prove that the numbers $N(k, q)$ are reconstructible. If $k > 1$ then $N(k, q)$ is reconstructible by Theorem 6(b), while $N(1, q) = n_{\rm c}(q)$ and is reconstructible by Theorem 8. //

Corollary 10.1. The flow, tension and chromatic polynomials and the chromatic number $\chi(G)$ of G are reconstructible.

Corollary 10.2. The class of perfect graphs is recognizable.

Proof. Both the clique number $\omega(G)$ (by Theorem 3) and $\chi(G)$ are reconstructible, hence so is the property '$\omega(G) = \chi(G)$'. And G is perfect iff $\omega(G) = \chi(G)$ and every graph G_u is perfect. //

Theorem 11. The characteristic polynomial of G is reconstructible.

Proof. Let A be an adjacency matrix of G, with characteristic polynomial

$$\begin{aligned} &x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \\ &= (x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_n) \end{aligned}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Let $\Sigma_k := \sum_{i=1}^n \lambda_i^k$. Then

$$\begin{aligned} a_{n-1} &= -\sum \lambda_i = -\Sigma_1 \ (=0), \\ a_{n-2} &= +\sum_{i<j} \lambda_i\lambda_j = \tfrac{1}{2}(\Sigma_1^2 - \Sigma_2), \\ a_{n-3} &= -\sum_{i<j<k} \lambda_i\lambda_j\lambda_k = -\tfrac{1}{6}(\Sigma_1^3 - 3\Sigma_1\Sigma_2 + 2\Sigma_3), \end{aligned}$$

etc, so it suffices to prove that $\Sigma_1, \ldots, \Sigma_n$ are reconstructible. But (even for $k > n$) $\Sigma_k = \text{trace } A^k = W_k$, the number of closed walks of length k in G. If $k \leqslant n$, a closed walk of length k either uses fewer than n vertices, or follows a hamiltonian circuit of G, and so the number of closed walks is reconstructible by Kelly's Lemma and Corollary 8.1. //

4. Generalizations. The *edge-reconstruction conjecture* is that every simple graph G with at least four edges is *edge-reconstructible*, that is, uniquely determined by the set of graphs $\{G - e : e \in E(G)\}$. There are counterexamples for graphs with two or three edges:

Theorem 12. The edge-reconstruction conjecture is true in general if it is true for all graphs without isolated vertices.

Proof. Let $|E(G)| = m \geqslant 4$. Let $t_k(G)$ denote the number of trails of length k in G, let $i(G)$ denote the number of isolated vertices of G, and let $i := \min_{e \in E(G)} i(G-e)$. By an obvious analogue of Kelly's Lemma, if $k < m$ then $t_k(G) = \frac{1}{m-k}\sum_{e\in E(G)} t_k(G-e)$ and so $t_k(G)$ is edge-reconstructible. The values of $t_2(G)$ and $t_3(G)$ determine whether a longest trail in G has

$$\begin{array}{rll} & \text{length } 1, & \text{when } \ i(G) = i-2, \\ & \text{length } 2, & \text{when } \ i(G) = i-1, \\ \text{or} & \text{length} \geqslant 3, & \text{when } \ i(G) = i. \end{array}$$

Thus $i(G)$ is edge-reconstructible. Now remove $i(G)$ isolated vertices from every graph $G - e$ to obtain the edge-deleted subgraphs of $G - i(G)K_1$, which has no isolated vertices. If this graph is edge-reconstructible, then so is G. //

Theorem 13. If $|E(G)| \geqslant 4$ and G is reconstructible and has no isolated vertices, then G is edge-reconstructible.

Proof. By the proof of Theorem 12, if $|E(G)| = m \geqslant 4$ we can recognize that G has no isolated vertices. We now construct its vertex-deleted subgraphs by the argument of Theorem 2. List all the vertex-deleted subgraphs of all the graphs $G - e$. For each $G_u \in D(G)$, the list contains $m - |E(G_u)|$ copies of G_u and one copy of every edge-deleted subgraph of G_u: call these the graphs *attributable* to G_u. (Note that $m - |E(G_u)| > 0$, since G has no isolated vertices.) Choose a graph in the list with most edges; it must be in $D(G)$, so call it G_{u_1}. Delete from the list all graphs attributable to G_{u_1}. Repeat until the list is empty. The graphs $G_{u_1}, \ldots, G_{u_n}$ so formed make up $D(G)$. If G is reconstructible, it is now determined. //

Theorems 12 and 13 show that the reconstruction conjecture implies the edge-reconstruction conjecture. V. Müller (1977) proved that G is edge-reconstructible if $2^{m-1} > n!$ (where $m = |E(G)|$, $n = |V(G)|$), hence if $m \geqslant n \log_2 n$ (which implies $2^m \geqslant n^n > 2n!$). So almost all graphs are edge-reconstructible.

There are analogous questions for digraphs, but there are so many counterexamples that few nice conjectures remain. Much the same is true for infinite graphs, although it is not known whether the reconstruction conjecture holds for locally finite connected infinite graphs, or for infinite trees. There is also the following conjecture.

Conjecture. If $G \sim H$ then each of G, H is isomorphic to an induced subgraph of the other.