Restriction and Contraction of Matroids

Theorem 1. Let $\boldsymbol{M} = (S, \boldsymbol{I})$ be a matroid and let $A \subseteq S$.

(a) Let $\boldsymbol{I}|A := \{X \in \boldsymbol{I} : X \subseteq A\} = \{X \cap A : X \in \boldsymbol{I}\}$. Then $\boldsymbol{M}|A := (A, \boldsymbol{I}|A)$ is a matroid on A of rank $r(A)$. It is called the **restriction** of $\boldsymbol{M}$ to A.

(b) Let B be a base of $S \setminus A$ and let $\boldsymbol{I}\cdot A := \{X \subseteq A : X \cup B \in \boldsymbol{I}\}$. Then $\boldsymbol{M}\cdot A := (A, \boldsymbol{I}\cdot A)$ is a matroid on A of rank $r(S) - r(S \setminus A)$, with rank function r' given by

$$r'(X) = r(X \cup B) - r(B) = r(X \cup (S \setminus A)) - r(S \setminus A)$$

(so that it is independent of the choice of B). It is called the **contraction** of $\boldsymbol{M}$ to A.

Proof. (a) It is easy to see that $\boldsymbol{I}|A$ is an independence structure, since removing all the independent sets not contained in A does not affect the truth of any of the independent-set axioms for those independent sets that remain. Thus $\boldsymbol{M}|A$ is a matroid.

(b) The fact that $\boldsymbol{M}\cdot A$ is a matroid can be deduced from the duality result established in the next theorem, since $\boldsymbol{M}\cdot A = (\boldsymbol{M}^*|A)^*$. To prove it directly, we check the independent-set axioms. Clearly $\varnothing \in \boldsymbol{I}\cdot A$ and any subset of a set in $\boldsymbol{I}\cdot A$ is in $\boldsymbol{I}\cdot A$. To prove the exchange property, let X and Y be sets in $\boldsymbol{I}\cdot A$ with $|Y| > |X|$. Then $X \cup B$ and $Y \cup B \in \boldsymbol{I}$ and $|Y \cup B| > |X \cup B|$, and so by the exchange property for $\boldsymbol{I}$ there exists a $y \in (Y \cup B) \setminus (X \cup B) = Y \setminus X$ such that $X \cup B \cup \{y\} \in \boldsymbol{I}$, that is, $X \cup \{y\} \in \boldsymbol{I}\cdot A$. Thus the exchange property holds and $\boldsymbol{M}\cdot A$ is a matroid.

To prove the formula for the rank function r' of $\boldsymbol{M}\cdot A$, let $X \subseteq A$ and let Y be a maximal subset of X in $\boldsymbol{I}\cdot A$. Then $Y \cup B \in \boldsymbol{I}$ by the definition of $\boldsymbol{I}\cdot A$. Moreover, $Y \cup B$ is a base both of $X \cup B$ and of $X \cup (S \setminus A)$: for, if not, then we could extend it to a base B' of $X \cup B$ or of $X \cup (S \setminus A)$ as appropriate, which could not contain any element of $(S \setminus A) \setminus B$ since B was already a maximal independent subset of $S \setminus A$, and so $B' = Y' \cup B$ where Y' properly contains Y; and this would contradict the maximality of Y, since Y' would be a larger subset of X in $\boldsymbol{I}\cdot A$. It follows that $Y \cup B$ is a base both of $X \cup B$ and of $X \cup (S \setminus A)$, and of course $r(B) = |B|$, so that $r'(X) = |Y| = r(X \cup B) - r(B) = r(X \cup (S \setminus A)) - r(S \setminus A)$. Thus $r'(X)$ does not depend on the choice of B as a base of $S \setminus A$, and nor does $\boldsymbol{M}\cdot A$, since $X \in \boldsymbol{I}\cdot A$ if and only if $r'(X) = |X|$. The rank of $\boldsymbol{M}\cdot A$ is $r'(S) = r(S \cup (S \setminus A)) - r(S \setminus A) = r(S) - r(S \setminus A)$. □

Theorem 2. (a) Restriction and contraction are dual operations, in the sense that $(\boldsymbol{M}|A)^* = \boldsymbol{M}^*\cdot A$ and $(\boldsymbol{M}\cdot A)^* = \boldsymbol{M}^*|A$.

(b) Restriction and contraction commute, in the sense that if A and B are disjoint subsets of S then $(\boldsymbol{M}|(S \setminus A))\cdot(S \setminus (A \cup B)) = (\boldsymbol{M}\cdot(S \setminus B))|(S \setminus (A \cup B))$. A matroid formed from $\boldsymbol{M}$ by restriction and contraction in this way is called a **minor** of $\boldsymbol{M}$.

Proof. (a) It follows from the definitions of $\boldsymbol{M}|A$ and $\boldsymbol{M}\cdot A$ that the bases of $\boldsymbol{M}|A$ are the maximal $\boldsymbol{M}$-independent subsets of A, which are the sets of the form $B \cap A$ such that B is a base of $\boldsymbol{M}$ for which $|B \cap A|$ is as large as possible; and the bases of $\boldsymbol{M}\cdot A$ are the sets of the form $B \cap A$ such that B is a base of $\boldsymbol{M}$ for which $|B \setminus A|$

is as large as possible. But B is a base of $\boldsymbol{M}$ for which $|B \cap A|$ is as large as possible if and only if $B^* := S \setminus B$ is a base of $\boldsymbol{M}^*$ for which $|B^* \setminus A|$ is as large as possible. It follows that the bases of $\boldsymbol{M}|A$ are the complements of the bases of $\boldsymbol{M}^* \cdot A$, so that $(\boldsymbol{M}|A)^* = \boldsymbol{M}^* \cdot A$. Applying this result to $\boldsymbol{M}^*$ gives $(\boldsymbol{M}^*|A)^* = \boldsymbol{M} \cdot A$, or $(\boldsymbol{M} \cdot A)^* = \boldsymbol{M}^*|A$.

(b) Let $X \subseteq S \setminus (A \cup B)$. Then $X \in (\boldsymbol{I}|(S \setminus A)) \cdot (S \setminus (A \cup B))$ if and only if there is a base B' of B such that $X \cup B' \in \boldsymbol{I}|(S \setminus A)$, which is the same as saying that $X \cup B' \in \boldsymbol{I}$. And $X \in (\boldsymbol{I} \cdot (S \setminus B))|(S \setminus (A \cup B))$ if and only if $X \in \boldsymbol{I} \cdot (S \setminus B)$, which means exactly the same. So

$$(\boldsymbol{M}|(S \setminus A)) \cdot (S \setminus (A \cup B)) = (\boldsymbol{M} \cdot (S \setminus B))|(S \setminus (A \cup B)). \quad \square$$

Theorem 3. (a) The circuits of $\boldsymbol{M}|A$ are the circuits of $\boldsymbol{M}$ that are contained in A, and the cocircuits of $\boldsymbol{M} \cdot A$ are the cocircuits of $\boldsymbol{M}$ that are contained in A.

(b) If $X \subseteq A$ and C is a circuit of $\boldsymbol{M}$, then there is base B of $S \setminus A$ such that $X \cup B$ contains C, if and only if $\varnothing \neq C \cap A \subseteq X$. Thus the circuits of $\boldsymbol{M} \cdot A$ are the minimal nonempty sets in the collection $\{C \cap A : C \text{ is a circuit of } \boldsymbol{M}\}$.

(c) Restriction and contraction of matroids correspond to deletion and contraction of edges in graphs, in the following sense. If $\boldsymbol{M}$ is the circuit matroid of a graph G and $A \subseteq E(G)$, then $\boldsymbol{M}|A$ is the circuit matroid of the graph obtained by deleting all edges of G not in A, and $\boldsymbol{M} \cdot A$ is the circuit matroid of the graph obtained by contracting all edges of G not in A. It follows that the minors of $\boldsymbol{M}$ are the circuit matroids of the subcontractions of G.

Proof. (a) Since a subset of A is independent in $\boldsymbol{M}|A$ if and only if it is independent in $\boldsymbol{M}$, it follows that the minimal $(\boldsymbol{M}|A)$-dependent subsets of A are the same as the minimal $\boldsymbol{M}$-dependent subsets; that is, the circuits of $\boldsymbol{M}|A$ are the circuits of $\boldsymbol{M}$ that are contained in A. Therefore, using the duality result from the previous theorem, the cocircuits of $\boldsymbol{M} \cdot A$, which are the circuits of $\boldsymbol{M}^*|A$, are the circuits of $\boldsymbol{M}^*$ that are contained in A, and hence are the cocircuits of $\boldsymbol{M}$ that are contained in A.

(b) Suppose $X \subseteq A$ and C is a circuit of $\boldsymbol{M}$. If there is a base B of $S \setminus A$ such that $C \subseteq X \cup B$, then clearly $\varnothing \neq C \cap A \subseteq X$, since $C \nsubseteq B$. Conversely, if $\varnothing \neq C \cap A \subseteq X$, then $C \setminus A$ is independent and so can be extended to a base B of $S \setminus A$ such that $C \subseteq X \cup B$. But X is dependent in $\boldsymbol{M} \cdot A$ if and only if $X \cup B$ is dependent in $\boldsymbol{M}$ (for *some* base B of $S \setminus A$, or, equivalently, for *every* base B of $S \setminus A$), which means that $X \cup B$ contains a circuit C of $\boldsymbol{M}$ and hence, by what we have just proved, that X contains the nonempty set $C \cap A$. The same argument shows that if $X = C \cap A \neq \varnothing$ then X is dependent in $\boldsymbol{M} \cdot A$. Thus the circuits of $\boldsymbol{M} \cdot A$ (the *minimal* dependent sets of $\boldsymbol{M} \cdot A$) are precisely the nonempty sets of the form $C \cap A$ such that C is a circuit of $\boldsymbol{M}$.

(c) Now suppose $\boldsymbol{M}$ is the circuit matroid of a graph G, $A \subseteq E(G)$ and $C \subseteq A$. For convenience, let $\bar{A} := E(G) \setminus A$. Then C is a circuit of the graph $G - \bar{A}$, obtained by deleting all edges of $\bar{A}$, if and only if C is a circuit in G. It follows from (a) that $\boldsymbol{M}(G - \bar{A}) = \boldsymbol{M}|A$. Also, C is a circuit of the graph $G/\bar{A}$, obtained by contracting all edges of $\bar{A}$, if and only if C is of the form $C' \cap A$, for some circuit C' of G. It follows from (b) that $\boldsymbol{M}(G/\bar{A}) = \boldsymbol{M} \cdot A$. Hence restriction and contraction of matroids correspond to deletion and contraction of edges in graphs. $\square$