Restricted Colourings of Graphs

1. Vertex-colourings. If $c : V \to \{1, \dots, k\}$ is an improper vertex-k-colouring of a finite graph $G = (V, E)$, let $d_i(v)$ denote the number of neighbours of vertex v with colour i, and let $d(v)$ denote the degree of v. Call an edge *bad* if its endvertices have the same colour.

Theorem 1.1. (First version, O. V. Borodin and A. V. Kostochka, 1977.) Let $f_1, f_2 : V \to \mathbb{Z}$ satisfy $f_1(v) + f_2(v) \geqslant d(v) - 1$, $\forall\, v \in V$. Then there exists a 2-colouring $c : V \to \{1, 2\}$ such that $d_{c(v)}(v) \leqslant f_{c(v)}(v)$, $\forall\, v \in V$.

Exercise. Show that $f_1(v) + f_2(v) \geqslant d(v) - 2$ is not enough.

Proof. Choose a 2-colouring c that minimizes

$$N_c := B_c - \tfrac{1}{2} \textstyle\sum_{v \in V} \left(f_{c(v)}(v) - f_{3-c(v)}(v)\right),$$

where B_c is the number of bad edges. If $d_{c(v)}(v) > f_{c(v)}(v)$ for some $v \in V$, change the colour of v from $c(v)$ to $3 - c(v)$. Then B_c increases by $d(v) - 2d_{c(v)}(v)$, and so N_c increases by

$$\begin{aligned} & d(v) - 2d_{c(v)}(v) + f_{c(v)}(v) - f_{3-c(v)}(v) \\ \leqslant\ & d(v) - 2d_{c(v)}(v) + 2f_{c(v)}(v) - d(v) + 1 \\ =\ & 2[f_{c(v)}(v) - d_{c(v)}(v)] + 1 \\ <\ & 0, \ \Rightarrow\Leftarrow. \end{aligned}$$

Therefore $d_{c(v)}(v) \leqslant f_{c(v)}(v)$ for all $v \in V$. //

Corollary 1.1.1. Let $f_1, \dots, f_k : V \to \mathbb{Z}$ satisfy $\sum_i f_i(v) \geqslant d(v) - k + 1$, $\forall\, v \in V$, where $k \geqslant 1$. Then there exists a k-colouring $c : V \to \{1, \dots, k\}$ such that $d_{c(v)}(v) \leqslant f_{c(v)}(v)$, $\forall\, v \in V$.

Proof. Exercise. (Induction on k.) //

Theorem 1.1. (Second version, C. Bernardi, DM 64 (1987) 95–96.) Let $f_1, f_2 : V \to \mathbb{R}$ satisfy $f_1(v) + f_2(v) > d(v)$, $\forall\, v \in V$. Then there exists a 2-colouring $c : V \to \{1, 2\}$ such that $d_{c(v)}(v) < f_{c(v)}(v)$, $\forall\, v \in V$.

Proof. W.l.o.g. $f_1(v)$ and $f_2(v)$ are integers whose sum is $d(v) + 1$. Let $m(v) := \max\{f_1(v), f_2(v)\}$. If $f_1(v) > f_2(v)$, join v to $f_1(v) - f_2(v) = m(v) - f_2(v)$ new independent vertices *permanently* coloured 2; and analogously if $f_2(v) > f_1(v)$. Do this for all $v \in V$ to form a new graph G' in which each (old) v has degree $2m(v) - 1$, and choose a 2-colouring $c : V \to \{1, 2\}$. Now,

$$\begin{aligned} & v \text{ is adjacent to } < f_{c(v)}(v) \text{ vertices of colour } c(v) \text{ in } G \\ \Longleftrightarrow\ & v \text{ is adjacent to } < m(v) \text{ vertices of colour } c(v) \text{ in } G'. \end{aligned}$$

If this is not so, change the colour of v and it will be so. Since this change reduces the number of bad edges in G', a finite number of repetitions will achieve the desired result. //

Exercise. Prove that the two versions of Theorem 1.1 are equivalent.
Exercise. State and prove the second version of Corollary 1.1.1.

Theorem 1.2. (R. H. Cowen and W. Emerson, unpublished?) Let $p_1, \dots, p_k$ be such that $0 \leqslant p_i \leqslant 1$, $\forall\, i$, and $\sum_{i=1}^k p_i \geqslant 1$. Then there is a k-colouring $c : V \to \{1, \dots, k\}$ s.t. $d_{c(v)}(v) \leqslant p_{c(v)} d(v)$ for each $v \in V$.

Exercise. Show that this does not follow if $\sum_{i=1}^k p_i < 1$.
Exercise. Derive Theorem 1.2 from Theorem 1.1.

Proof. Choose a k-colouring c that minimizes $N_c := \sum_i \frac{b_i}{p_i}$, where b_i is the number of edges with both ends coloured i. If v has colour i and $d_i(v) > p_i d(v)$, then $\exists\, j$ s.t. $d_j(v) < p_j d(v)$. Change the colour of v from i to j. N_c goes down by $\frac{d_i(v)}{p_i} - \frac{d_j(v)}{p_j} > d(v) - d(v) = 0$, $\Rightarrow\Leftarrow$. //

2. Edge-colourings. Let $G = (V, E)$ as before. If $c : E \to \{1, \dots, k\}$ is an improper edge-k-colouring of G, let $d_i(v)$ now denote the number of edges at v with colour i ($i = 1, \dots, k$; $v \in V$).

Conjecture. (D. R. Woodall, 1985/1990.) Let $p_1, \dots, p_k$ be such that $0 \leqslant p_i \leqslant 1$, $\forall\, i$, and $\sum_{i=1}^k p_i \geqslant 1$. Then there is a k-colouring $c : E \to \{1, \dots, k\}$ s.t.

$$d_i(v) \leqslant p_i d(v) + 1 \qquad (*)$$

($i = 1, \dots, k$; $v \in V$).

Exercise. Show that this would imply Vizing's theorem, that $\chi'(G) \leqslant \Delta + 1$.

Exercise. Show that the 1 in $(*)$ cannot be replaced by any *constant* less than 1. (Hint: Not every graph is edge-Δ-colourable.)

Theorem 2.1. The conjecture is true if $k = 2$.

Proof. Let G be a minimal counterexample, and $e = uw \in E$, so that $G - e$ has such a 2-colouring. Give e colour 1. If now $d_1(w) > p_1 d(w) + 1$ then

$$d_2(w) = d(w) - d_1(w) < (1 - p_1)d(w) - 1 \leqslant p_2 d(w) - 1,$$

so give e colour 2 instead. Then u is the only vertex where $(*)$ fails.

Let $V_i := \{v \in V : d_i(v) > p_i d(v)\}$ ($i = 1, 2$). Clearly $V_1 \cap V_2 = \emptyset$. W.l.o.g. $u \in V_1$, so that $p_1 d(u) + 1 < d_1(u) \leqslant p_1 d(u) + 2$.

Let H be the union of all trails

$$u = v_1,\ e_1,\ v_2,\ e_2,\ \dots$$

in which edge e_j has colour 1 (2) if j is odd (even). In each such trail T, $v_j \in V_1$ (V_2) if j is odd (even), since if v_j were the first vertex along T for which this failed, then changing the colours of $e_1, \dots, e_{j-1}$

would satisfy $(*)$ everywhere, $\Rightarrow\Leftarrow$. Thus H is bipartite on two sets $V_1 \cap H$ and $V_2 \cap H$.

Note that H contains every edge of G that has an end in $V_i \cap H$ and is coloured i ($i = 1, 2$). Thus the *proportion* of edges of H that have colour i is $> p_i$. But this cannot hold for $i = 1, 2$ simultaneously, $\Rightarrow\Leftarrow$. //

Exercise. Use this theorem and induction on k to prove the weaker version of the conjecture with $(*)$ replaced by $d_i(v) \leqslant p_i d(v) + \lceil \log_2 k \rceil$.

Theorem 2.2. (R. O. Davies, unpublished?) Let $G = (U, W, E)$ be a bipartite graph and let $f : U \to \mathbb{Z}$ and $g : W \to \mathbb{Z}$ be functions with nonnegative integer values. Then there exists an edge-2-colouring $c : E \to \{\text{blue}, \text{red}\}$ such that $d_{\text{blue}}(u) \leqslant f(u)\ \forall\, u \in U$ and $d_{\text{red}}(w) \leqslant g(w)\ \forall\, w \in W$, if and only if, for each pair of subsets $I \subseteq U$ and $J \subseteq W$, the number of edges of G between I and J is at most $\sum_{u \in I} f(u) + \sum_{w \in J} g(w)$.

Proof. 'Only if' is clear, and so we proof 'if'. For each $u \in U$, $w \in W$, let $u_1, \dots, u_{f(u)}$ be copies of u and $w_1, \dots, w_{g(w)}$ be copies of w. For each edge $e = uw \in E$, let $X_e := \{u_1, \dots, u_{f(u)}, w_1, \dots, w_{g(w)}\}$. For each subset E' of E, let $I(E')$ and $J(E')$ be the sets of endvertices of edges of E' in U and W respectively. Then

$$\Big| \bigcup_{e \in E'} X_e \Big| = \sum_{u \in I(E')} f(u) + \sum_{w \in J(E')} g(w) \geqslant |E'|.$$

It follows from Hall's theorem that $(X_e : e \in E)$ has a system of distinct representatives. For each edge $e = uw$, colour e blue if the representative x_e of X_e is u_k for some k, and red if x_e is w_k for some k. Then at most $f(u)$ edges at u are blue and at most $g(w)$ edges at w are red. //