The Tutte Polynomial

Let $G = (V, E)$ be a graph with n vertices, m edges and c components, possibly containing loops and multiple edges. If $X \subseteq E$, let $G(X)$ denote the graph (V, X), let $c(X) = c_G(X)$ be the number of components of $G(X)$, and let the *rank* of X be $r(X) = r_G(X) := n - c_G(X)$, the number of edges in a skeleton (maximal subforest) of $G(X)$. Note that

$$r_G(E) - r_G(X) = c_G(X) - c_G(E) = c_G(X) - c,$$

and $|X| - r_G(X)$ is equal to the *corank* or *circuit rank* $\gamma(X) = \gamma_G(X)$ of X. Thus

$$\gamma(X) - c(X) = |X| - r(X) + r(X) - n = |X| - n.$$

The *Tutte polynomial* $T(G, x, y)$ of G (W. T. Tutte, 1947) is defined by

$$T(G, x, y) := \sum_{X \subseteq E} (x-1)^{c_G(X) - c_G(E)} (y-1)^{\gamma_G(X)}.$$

Theorem 1. If $e \in E$ then

$$T(G,x,y) = \begin{cases} xT(G/e, x, y) & \text{if } e \text{ is a cutedge,} \\ yT(G-e, x, y) & \text{if } e \text{ is a loop,} \\ T(G-e, x, y) + T(G/e, x, y) & \text{otherwise.} \end{cases}$$

Proof. Suppose $X \subseteq E$ and $e \in E \setminus X$.

If e is a cutedge:	$c_G(X) = c_{G/e}(X) + 1$,	$\gamma_G(X) = \gamma_{G/e}(X)$,
otherwise (in fact always):	$c_G(X) = c_{G-e}(X)$,	$\gamma_G(X) = \gamma_{G-e}(X)$.
If e is a loop:	$c_G(X \cup \{e\}) = c_{G-e}(X)$,	$\gamma_G(X \cup \{e\}) = \gamma_{G-e}(X) + 1$,
otherwise:	$c_G(X \cup \{e\}) = c_{G/e}(X)$,	$\gamma_G(X \cup \{e\}) = \gamma_{G/e}(X)$.

Also $c_G(E) = c_{G/e}(E \setminus \{e\}), = c_{G-e}(E \setminus \{e\})$ if e is not a cutedge.

For brevity let $f_G(X) := (x-1)^{c_G(X) - c_G(E)}(y-1)^{\gamma_G(X)}$. Then

$$f_G(X) = \begin{cases} (x-1) f_{G/e}(X) & \text{if } e \text{ is a cutedge,} \\ f_{G-e}(X) & \text{otherwise,} \end{cases}$$

and

$$f_G(X \cup \{e\}) = \begin{cases} (y-1) f_{G-e}(X) & \text{if } e \text{ is a loop,} \\ f_{G/e}(X) & \text{otherwise.} \end{cases}$$

Thus

$$f_G(X) + f_G(X \cup \{e\}) = \begin{cases} x f_{G/e}(X) & \text{if } e \text{ is a cutedge,} \\ y f_{G-e}(X) & \text{if } e \text{ is a loop,} \\ f_{G-e}(X) + f_{G/e}(X) & \text{otherwise.} \end{cases}$$

But

$$T(G, x, y) = \sum_{\substack{X \subseteq E \\ e \notin X}} \big(f_G(X) + f_G(X \cup \{e\}) \big),$$

$$T(G-e, x, y) = \sum_{X \subseteq E(G-e)} f_{G-e}(X),$$

and

$$T(G/e, x, y) = \sum_{X \subseteq E(G/e)} f_{G/e}(X),$$

and so the result follows. //

Exercise. $T(G, x, y)$ is multiplicative over blocks; that is, if G has blocks $B_1, \ldots, B_b$, then $T(G, x, y) = \prod_{i=1}^{b} T(B_i, x, y)$.

Examples.

G	$T(G,x,y)$
$nK_1 = \bar{K}_n$	1
Any tree	x^m
m loops: 	y^m
C_m	$y+x+x^2+\ldots+x^{m-1}$
m || edges: 	$x+y+y^2+\ldots+y^{m-1}$

Theorem 2. If G is planar and G^* is a dual of G, then $T(G,x,y) = T(G^*,y,x)$.

Proof by induction on m, using Theorem 1. Note that if e^* is the edge of G^* corresponding to the edge e of G, then e^* is a loop (cutedge) iff e is a cutedge (loop), and $G^* - e^*$ and G^*/e^* are duals of G/e and $G - e$ respectively. //

Theorem 3.

(a) $(-1)^{n-c}t^c T(G,1-t,0) = P(G,t)$, the chromatic polynomial of G.

(b) $(-1)^{n-c}T(G,1-t,0) = \widetilde{T}(G,t)$, the tension polynomial of G.

(c) $(-1)^{m-n+c}T(G,0,1-t) = F(G,t)$, the flow polynomial of G.

Proof by induction on m. If $m = 0$ (and $n = c$) then $T(\bar{K}_c,x,y) = \widetilde{T}(\bar{K}_c,t) = F(\bar{K}_c,t) = 1$ and $P(\bar{K}_c,t) = t^c$, and so the results all hold. If $m > 0$ and $e \in E$ then

$$\begin{aligned}
P(G,t) &= P(G-e,t) - P(G/e,t), \\
\widetilde{T}(G,t) &= \widetilde{T}(G-e,t) - \widetilde{T}(G/e,t) \quad \text{if } e \text{ is not a cutedge,} \\
F(G,t) &= F(G/e,t) - F(G-e,t) \quad \text{if } e \text{ is not a loop.}
\end{aligned}$$

If e is a loop, then

$$\begin{aligned}
&P(G,t) = \widetilde{T}(G,t) = T(G,1-t,0) = 0, \\
&T(G,0,1-t) = (1-t)T(G-e,0,1-t) \quad \text{by Theorem 1,} \quad (1) \\
&F(G,t) = (t-1)F(G-e,t) \quad \text{(clearly)} \\
&\qquad = (t-1)(-1)^{m-1-n+c}T(G-e,0,1-t) \quad \text{by induction,} \\
&\qquad = (-1)^{m-n+c}T(G,0,1-t) \quad \text{by (1).}
\end{aligned}$$

If e is a cutedge, then

$$\begin{aligned}
&F(G,t) = T(G,0,1-t) = 0, \\
&T(G,1-t,0) = (1-t)T(G/e,1-t,0) \quad \text{by Theorem 1,} \quad (2)
\end{aligned}$$

$$\begin{aligned}
\widetilde{T}(G,t) &= (t-1)\widetilde{T}(G/e,t) \quad \text{(clearly)} \\
&= (t-1)(-1)^{n-1-c}T(G/e,1-t,0) \quad \text{by induction,} \\
&= (-1)^{n-c}T(G,1-t,0) \quad \text{by (2),} \\
P(G,t) &= (t-1)P(G/e,t) = (t-1)t^c\widetilde{T}(G/e,t) = t^c\widetilde{T}(G,t).
\end{aligned}$$

Otherwise

$$\begin{aligned}
\widetilde{T}(G,t) &= \widetilde{T}(G-e,t) - \widetilde{T}(G/e,t) \\
&= (-1)^{n-c}T(G-e,1-t,0) - (-1)^{n-1-c}T(G/e,1-t,0) \\
&\qquad \text{by induction,} \\
&= (-1)^{n-c}T(G,1-t,0) \quad \text{by Theorem 1,}
\end{aligned}$$

etc. //

Theorem 4. (a) $T(G,1+t^{-1},1+t) = t^{c-n}(1+t)^m$. (This gives the value of $T(G,x,y)$ on the hyperbola $(x-1)(y-1) = 1$.)
(b) $T(G,2,2) = 2^m$.
(c) $T(G,0,0) = 0$ if $m \neq 0$.
(d) $T(G,1,1) =$ number of skeletons of G.
(e) $T(G,2,1) =$ number of circuit-free subsets of E.
(f) $T(G,1,2) =$ number of subsets of E containing a skeleton of G.

Proof. (a)

$$\begin{aligned}
T(G,1+t^{-1},1+t) &= \sum_{X \subseteq E} t^{-(r_G(E)-r_G(X))}t^{|X|-r_G(X)} \\
&= t^{-r_G(E)} \sum_{X \subseteq E} t^{|X|} \\
&= t^{c-n}(1+t)^m.
\end{aligned}$$

(b) $T(G,2,2) = \sum_{X \subseteq E} 1 = 2^m$.

(c) $T(G,0,0) = (-1)^{c-n}(1-1)^m = 0$ (putting $t = -1$ in (a)).

(d) $T(G,1,1) = \sum_{X \subseteq E} 0^{r_G(E)-r_G(X)}0^{|X|-r_G(X)}$

which (since $0^0 = 1$) is the number of $X \subseteq E$ such that $r_G(E) = r_G(X) = |X|$. But $r_G(E) = r_G(X)$ iff X contains a skeleton of G, $r_G(X) = |X|$ iff X is circuit-free, and both hold simultaneously iff (V, X) is a skeleton of G.

(e) $T(G, 2, 1) = \sum_{X \subseteq E} 0^{|X| - r_G(X)}$.

(f) $T(G, 1, 2) = \sum_{X \subseteq E} 0^{r_G(E) - r_G(X)}$. //