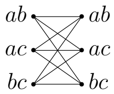
Vertex-Colourings and List Colourings

1. Choosability of planar graphs. Suppose $G = (V, E)$ is a graph and each vertex $v \in V$ is given a 'list' (set) $L(v)$ of 'colours'. An *L-colouring* of G is a vertex-colouring in which each vertex v receives a colour $c(v) \in L(v)$; we talk of *colouring G from its lists*. If $f : V \to \mathbb{N}$ is a function, then G is *f-choosable* if G is properly L-colourable whenever $|L(v)| = f(v)$ (or, equivalently, $|L(v)| \geqslant f(v)$) $\forall\, v \in V$. G is *k-choosable* or *list-k-colourable* if G is f-choosable when $f(v) = k\ \forall\, v \in V$. The minimum k for which G is k-choosable is its *choosability* or *list chromatic number* $\mathrm{ch}(G)$. Clearly $\mathrm{ch}(G) \geqslant \chi(G)$, since if every vertex has the *same* list of $\mathrm{ch}(G)$ colours then a proper L-colouring is a proper $\mathrm{ch}(G)$-colouring of G.

Example. $\chi(K_{3,3}) = 2$ but $\mathrm{ch}(K_{3,3}) = 3$. $\geqslant 3$:

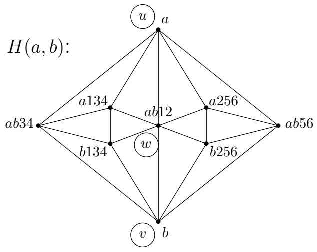


$\leqslant 3$: exercise.

By the same reasoning, $\mathrm{ch}(K_{r,r}) \geqslant k + 1$ if $r \geqslant \binom{2k-1}{k}$. Since $\chi(K_{r,r}) = 2$, this shows that there is no upper bound for $\mathrm{ch}(G)$ in terms of $\chi(G)$.

Theorem 1. (M. Voigt, 1993.) Not every planar graph is 4-choosable.

Proof. (S. Gutner, 1996.) The graph $H(a, b)$ shown cannot be coloured from its lists. (If w is coloured 1, we get a contradiction on the left; if 2, we get a contradiction on the right.) Now form G by



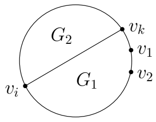
taking 12 copies of $H(a, b)$, where (a, b) runs through all ordered pairs of distinct elements of $\{7, 8, 9, 10\}$. Identify all 12 vertices labelled u and all 12 vertices labelled v, add an edge uv, and give u and v list $\{7, 8, 9, 10\}$. Clearly G is planar. Whatever colours a and b are given to u and v, we get a contradiction in $H(a, b)$. //

Note that G is even 3-colourable.

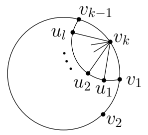
Theorem 2. (C. Thomassen, 1994.) Every planar graph is 5-choosable.

Proof. We shall prove the following stronger result. Let G be a 2-connected near-triangulation of the plane; i.e., the outside face is bounded by a circuit $C : v_1v_2 \dots v_kv_1$ and every inside face is a triangle. Suppose $L(v_1) = \{1\}$, $L(v_2) = \{2\}$, $|L(v_i)| \geqslant 3$ if $3 \leqslant i \leqslant k$, and $|L(v)| \geqslant 5$ if $v \in G - C$. Then G can be coloured from its lists. This will show that G, and hence every subgraph of G, is 5-choosable, and this will prove the theorem.

Proof of the stronger result. We use induction on $|V(G)|$. If $|V(G)| = 3$ then the result is obvious; so suppose $|V(G)| \geqslant 4$.



If C has a chord v_iv_k $(2 \leqslant i \leqslant k - 2)$, then G is the union of two near-triangulations G_1 and G_2 with bounding circuits $C_1 : v_1v_2 \dots v_iv_kv_1$ and $C_2 : v_iv_{i+1} \dots v_kv_i$ respectively. Apply the induction hypothesis first to extend the colouring of v_1, v_2 to G_1, and then to extend the resulting colouring of v_i, v_k to G_2. This gives the required L-colouring of G. So we may suppose that C has no such chord.



Let v_k have neighbours $v_1, u_1, u_2, \dots, u_l, v_{k-1}$ in clockwise order. Then G contains the path $P : v_1u_1u_2 \dots u_lv_{k-1}$, and $G - v_k$ is a near-triangulation bounded by the circuit

$C' := P \cup (C - v_k)$. Let x, y be distinct colours in $L(v_k) \setminus \{1\}$. Define $L'(u_i) := L(u_i) \setminus \{x, y\}$ $(i = 1, \dots, l)$ and $L'(v) := L(v)$ for all other vertices $v \in V(G - v_k)$, so that $|L'(v)| \geqslant 3$ if $v \in C' - \{v_1, v_2\}$ and $|L'(v)| \geqslant 5$ if $v \notin C'$. Apply the induction hypothesis to colour $G - v_k$ from the new lists L'. Finally, give colour x or y to v_k so that v_{k-1} and v_k get different colours. This gives a proper L-colouring of G, since none of the vertices $v_1, u_1, u_2, \dots, u_l$ has colour x or y. //

2. The block structure of a graph. A *cutvertex* or *cut-edge* in a graph is a vertex or edge whose removal disconnects the component in which it lies. A cut-edge is also called a *bridge*, *isthmus* or *coloop*. A graph G is *2-connected* if $|V(G)| \geqslant 3$ and G is connected and has no cutvertex.

Lemma 3.1. If $|V(G)| \geqslant 3$ then the following are equivalent.
(i) G is 2-connected.
(ii) $\forall\, v_1, v_2 \in V(G)$, there is a circuit in G containing v_1 and v_2.
(iii) $\forall\, e_1, e_2 \in E(G)$, there is a circuit in G containing e_1 and e_2, and G has no isolated vertices.

Proof. Exercise: (i) $\Longrightarrow$ (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i). (Assume Menger's theorem.) //

Lemma 3.2. Let G be a graph, $e \in E(G)$. Then the following are equivalent.
(i) G has a 2-connected subgraph that contains e.
(ii) There is a circuit in G that contains e.
(iii) e is not a cut-edge of G.

Proof. Exercise: (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii). //

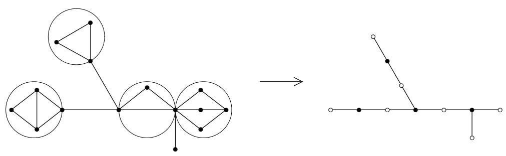
A connected graph G is *separable* if it has a cutvertex and *nonseparable* otherwise, i.e., if $G - v$ is connected $\forall\, v \in V(G)$ ($\Longleftrightarrow$ G is 2-connected or K_1 or K_2).

Lemma 3.3. Let G_1 and G_2 be nonseparable graphs with at least two vertices in common. Then $G_1 \cup G_2$ is nonseparable.

Proof. For every vertex v of $G_1 \cup G_2$, $G_1 - v$ and $G_2 - v$ are connected graphs with at least one vertex in common, and so $(G_1 \cup G_2) - v, = (G_1 - v) \cup (G_2 - v)$, is connected. //

A *block* of a graph G is a maximal nonseparable subgraph with at least one edge. So, by Lemma 3.2, it is either a maximal 2-connected subgraph of G, or a cut-edge and its two endvertices.

Theorem 3. Let G be a connected graph.
(a) Every edge of G is in exactly one block.
(b) The block structure of G is *tree-like* in the following sense: if a new graph T is defined to contain vertices $v(B)$ and $v(w)$ for every block B and cutvertex w of G, with an edge $v(B)v(w)$ iff $w \in B$, then T is a tree (called the *block-cutvertex tree* of G).



Proof. (a) An edge e is in at least one block since K_2 is nonseparable with at least one edge. If B_1 and B_2 are two blocks containing e then $B_1 \cup B_2$ is nonseparable by Lemma 3.3, and so $B_1 = B_1 \cup B_2 = B_2$ by the maximality of blocks. Thus e lies in exactly one block.

(b) If the block structure is not tree-like then there is a sequence of distinct blocks and cutvertices

$$B_1, w_1, B_2, w_2, \dots, B_r, w_r, B_{r+1} = B_1$$

where $\{w_i\} = B_i \cap B_{i+1}$ $(1 \leqslant i \leqslant r)$. But then there is a circuit C in G that contains at least two vertices of each of $B_1, \ldots, B_r$ but is not contained in any of them. Thus $B_1 \cup C$ is a nonseparable subgraph that properly contains B_1, $\Rightarrow\Leftarrow$. //

An *endblock* is a block that contains exactly one cutvertex.

Corollary 3.1. A connected graph that is separable has at least two endblocks.

Proof. The endblocks correspond to the endvertices of the tree, and every tree (except K_1) has at least two endvertices. //

3. Choosability of graphs in surfaces. We need the choosability analogue of Brooks's theorem.

Theorem 4. (R. L. Brooks, 1941.) If G is a connected graph that is not complete or an odd circuit then $\chi(G) \leqslant \Delta(G)$. //

Theorem 5. (V. G. Vizing, 1976; P. Erdős, A. L. Rubin and H. Taylor, 1979.) $\mathrm{ch}(G) \leqslant \Delta(G)$ whenever $\chi(G) \leqslant \Delta(G)$. More precisely, if G is connected and $|L(v)| = \Delta(G)$ for each $v \in V(G)$, then G can be coloured from its lists unless G is a complete graph or an odd circuit and L is constant.

Proof. We use induction on $|V(G)|$. The result is easy to see if $|V(G)| \leqslant 2$, so suppose $|V(G)| \geqslant 3$. Let $B := G$ and $v \in V(G)$ if G is 2-connected, otherwise let B be an endblock of G with cutvertex v.

Suppose first that L is not constant on B. Then there exist adjacent vertices u, w in B s.t. $L(u) \neq L(w)$ and $w \neq v$, so that $G - w$ is connected. Give w a colour in $L(w) \setminus L(u)$, then colour all other vertices of G so that the graph induced by the uncoloured vertices is always connected and u is coloured last. This is possible

since, at the moment of its colouring, each vertex except u has at most $\Delta(G) - 1$ coloured neighbours (and a list of $\Delta(G)$ colours); and u has a neighbour w with a colour not in $L(u)$.

So we may suppose that L is constant on B. If $G \neq B$ then $G = H \cup B$ where $H \cap B = \{v\}$. If H or B is complete or an odd circuit then it has maximum degree $< \Delta(G)$ and is easily coloured once v is coloured. Otherwise, H and/or B can be coloured by induction. Since L is constant on B, we can permute the colours on B so that v has the same colour in both colourings, giving a proper L-colouring of G. It remains to consider the case when $G = B$, so that L is constant on G; then the result follows from Brooks's theorem. //

A graph G is *k-degenerate* if it satisfies either of the following equivalent conditions.

(i) Every subgraph of G contains a vertex with degree (in the subgraph) at most k.

(ii) The vertices of G can be listed as $v_1, \ldots, v_n$ so that each vertex v_i has at most k neighbours v_j with $j < i$.

Clearly k-degenerate $\Longrightarrow$ $(k+1)$-choosable.

If S is a (compact connected) surface then $\chi(S)$ and $\mathrm{ch}(S)$ denote the maximum values of $\chi(G)$ and $\mathrm{ch}(G)$ taken over all graphs G that can be embedded in S.

Theorem 6. $\mathrm{ch}(S) = \chi(S)$ unless S is the sphere ($\cong$ plane).

Proof. P. J. Heawood (1890) defined a number $h(S)$ and proved that, except when S is the sphere, every graph in S is $(h(S) - 1)$-degenerate. G. Ringel and J. W. T. Youngs (1968) proved that, except when S is the Klein bottle, $K_{h(S)}$ can be embedded in S. It follows that

$$\chi(S) \leqslant \mathrm{ch}(S) \leqslant h(S) \leqslant \chi(S),$$

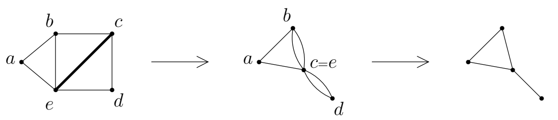
so that equality holds throughout. Heawood's argument also shows that every graph in the Klein bottle B *either* contains a vertex with degree $\leqslant 5$ *or* is 6-regular. P. Franklin (1934) showed that K_7 cannot be embedded in B (but K_6 can be), so that a 6-regular graph in B is not complete. A simple argument using Theorem 5 now shows that $\mathrm{ch}(B) = \chi(B) = 6$. //

$$h(S) = \left\lfloor \frac{7+\sqrt{49-24\epsilon(S)}}{2} \right\rfloor.$$

$\epsilon(S)$	$h(S)$	
2	4	sphere (plane)
1	6	projective plane
0	7	torus; Klein bottle
$\vdots$	$\vdots$	

$$\begin{gathered} |V| - |E| + |F| \geqslant \epsilon(S), \\ 2|E| \geqslant 3|F|, \\ 2|E| = a|V|, \\ \Longrightarrow (a-6)|V| + 6\epsilon(S) \leqslant 0. \\ \text{But } |V| \geqslant \Delta + 1 \geqslant a + 1. \\ \text{Assume } a \geqslant 6; \text{ then} \\ a^2 - 5a - 6 + 6\epsilon(S) \leqslant 0, \\ \Longrightarrow \delta \leqslant \lfloor a \rfloor \leqslant h(S) - 1. \end{gathered}$$

4. Minors. One *contracts* an edge in a graph by deleting it, then identifying its former endvertices and removing one edge of any pair of parallel edges so formed:



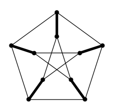
A *contraction* of a graph is any graph formed by contracting some of its edges, and a *subcontraction* or *minor* is a subgraph of a contraction or, equivalently, a contraction of a subgraph. Often we say that H is a minor of G is G has a minor *isomorphic* to H. So H is a minor of G iff there exist nonempty disjoint subsets V_x $(x \in V(H))$ of $V(G)$ such that

(i) each subset V_x induces a connected subgraph of G,

(ii) for each edge xy of H there is at least one edge of G between V_x and V_y.

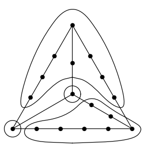
A graph is *H-minor-free* if it has no minor isomorphic to H.

Examples. (1) Every subgraph of a graph G is a minor of G.



(2) The Petersen graph has K_5 (and hence every graph with $\leqslant 5$ vertices) as a minor.

(3) Two graphs are *homeomorphic* if each can be obtained from the other by replacing edges by paths and *vice versa* (i.e., by inserting and suppressing vertices of degree 2). If $\delta(H) \geqslant 3$ and G has a subgraph homeomorphic to H, then H is a minor of G. E.g. $(H = K_4)$:



In fact (exercise?):

(i) if $\Delta(H) \leqslant 3$ and H is a minor of G then G has a subgraph homeomorphic to H;

(ii) if $\Delta(H) \leqslant 4$ and H is a minor of G then G has a subgraph homeomorphic to H or to a graph obtained from H by replacing one or more vertices of degree 4 by .

Theorem 7. (K. Wagner, 1937.) A graph is planar if and only if it does not have K_5 or $K_{3,3}$ as a minor.

Proof. Every minor of a planar graph is planar, and so

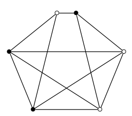
G has a subgraph homeomorphic to K_5 or $K_{3,3}$

$\Longrightarrow$ G has a K_5 or $K_{3,3}$ minor

$\Longrightarrow$ G is nonplanar.

The result follows from Kuratowski's theorem (1930). //

Note that many graphs have a K_5 minor but no subgraph homeomorphic to K_5:

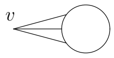


Theorem 8. Let G be a K_4-minor-free graph. Then

(a) G is not 3-connected;

(b) if $|V(G)| \geqslant 4$ then G has two nonadjacent vertices each with degree at most 2.

Proof. (a) Suppose G is 3-connected. Then it contains a nonhamiltonian circuit C (since any hamiltonian circuit must have a chord). If $v \in V(G) \setminus V(C)$ then there are three paths from v to $V(C)$ that are disjoint apart from the vertex v; thus G has a K_4 minor, $\Rightarrow\Leftarrow$.



(b) We prove the result by induction on $|V(G)|$. It holds if $|V(G)| = 4$ (any two nonadjacent vertices will do), so suppose $|V(G)| > 4$. W.l.o.g. G is connected. Let X be a smallest cutset of G, say $G = G_1 \cup G_2$ where $G_1 \cap G_2 = \langle X \rangle_G$, $|V(G_1)| > |X|$ and $|V(G_2)| > |X|$. By (a), $|X| \leqslant 2$. If $X = \{x, y\}$ and x, y are nonadjacent in G, let $H_i := G_i \cup \{xy\}$, otherwise let $H_i := G_i$ $(i = 1, 2)$. Then H_1 and H_2 are minors of G and so are K_4-minor-free. By the induction hypothesis, H_i has a vertex $v_i \notin X$ with degree $\leqslant 2$ (in H_i, and hence in G), and now v_1 and v_2 are two nonadjacent vertices with this property. //

Corollary 8.1. (G. A. Dirac, 1952.) If G is a graph with minimum degree $\delta(G) \geqslant 3$ then G has a K_4 minor. //

Note that if $\delta(G) \geqslant 2$ then G has a circuit, hence a K_3 minor. But the icosahedron is 5-regular with no K_5 minor, and so we need $\delta(G) \geqslant 6$ to force a K_5 minor.

Theorem 9. Wagner's Equivalence Theorem (WET) (1937). The 4-colour theorem (4CT) is equivalent to the statement that every K_5-minor-free graph is 4-colourable. //

Theorem 10. (N. Robertson and P. D. Seymour; announced 1985, manuscript 1996, published 2004.) Among any infinite collection of graphs there exist two, one of which is a minor of the other. //

Corollary 10.1. For fixed r, let $\mathcal{S}_r$ be the set of graphs G s.t. G is not r-colourable but every proper minor of G is r-colourable. Then $\mathcal{S}_r$ is finite. //

Note that if G has no minor in $\mathcal{S}_r$ then G is r-colourable. *Hadwiger's conjecture* is that $\mathcal{S}_r = \{K_{r+1}\}$.

5. Hadwiger's conjecture. Let $\mathcal{G}(H)$ denote the class of H-minor-free graphs. Hadwiger's conjecture (1943) is that every graph in $\mathcal{G}(K_{r+1})$ is r-colourable $(r = 1, 2, \ldots)$. For

$r = 1$ this is obvious, since $\mathcal{G}(K_2)$ comprises the edgeless graphs;

$r = 2$ easy, since $\mathcal{G}(K_3)$ comprises the forests;

$r = 3$ true by Corollary 8.1 (H. Hadwiger, 1943; G. A. Dirac, 1952);

$r = 4$ equivalent to the 4CT, by WET;

$r = 5$ equivalent to the 4CT, since N. Robertson, P. D. Seymour and R. Thomas proved in 1993 that if G is a non-5-colourable K_6-minor-free graph with as few vertices as possible, then $G - v$ is planar for some $v \in V(G)$;

$r \geqslant 6$ unknown but 'almost always' true (B. Bollobás, P. A. Catlin and P. Erdős, 1980).

Hadwiger's conjecture is equivalent to A + B:

A. Every graph in $\mathcal{G}(K_{r+1})$ has an independent set containing at least $\frac{1}{r}$ of its vertices.

B. If every minor of G has an independent set containing at least $\frac{1}{r}$ of its vertices, then G is r-colourable.

The next result goes 'half way' towards A.

Theorem 11. (D. R. Woodall, 1987.) If $r \geqslant 3$, $G \in \mathcal{G}(K_{r+1})$ and $v \in V(G)$, then G contains $r - 1$ (or fewer) disjoint independent sets of vertices whose union covers at least half the vertices of G including v. [$\geqslant \frac{1}{2}(n + r - 2)$ vertices if $n \geqslant r$]

Proof. We use induction on $n := |V(G)|$. The result is obvious if $n \leqslant r - 1$, so suppose $n \geqslant r$. We may assume that G is connected (otherwise apply induction to each component).

Case 1: $N(v) = V(G) \setminus \{v\}$. Then $G - v \in \mathcal{G}(K_r)$. Since $G - v$ is a forest if $r = 3$, and by induction if $r \geqslant 4$, $G - v$ contains $r - 2$ disjoint independent sets whose union contains at least $\frac{1}{2}(n-1)$ vertices. Add $\{v\}$ as the final independent set.

Case 2: $\exists\, w \in V(G) \setminus (N(v) \cup \{v\})$. Choose w at distance 2 from v; then $\exists\, u$ s.t. $vu, uw \in E(G)$ but $vw \notin E(G)$. Form G' from G by contracting u, v, w into a new vertex x. $G' \in \mathcal{G}(K_{r+1})$ and has $n - 2$ vertices. By induction, G' contains $r - 1$ disjoint independent sets covering at least $\frac{1}{2}(n - 2)$ vertices including x. Replace x by v and w to form the required independent sets in G. //

Corollary 11.1. Every graph in $\mathcal{G}(K_{r+1})$ has independence number $\alpha(G) \geqslant \frac{n}{2(r-1)}$. //

Duchet and Meyniel (1982) proved $\alpha(G) \geqslant \frac{n}{2r} + \frac{1}{2}$.

6. Improper colourings. A graph G is *k-colourable with defect d*, or *$(k, d)^*$-colourable*, if its vertices can be coloured with k colours so that no vertex has more than d neighbours with the same colour as itself; i.e., G contains no monochromatic $K_{1,d+1}$. So $(k, 0)^*$-colourable = (properly) k-colourable.

Theorem 12. (D. R. Woodall, 1990.) Suppose H is connected, $H \neq K_1$, and $G \in \mathcal{G}(K_1 + H)$. Then G can be (improperly)

2-coloured with no monochromatic H minor; i.e., so that no subgraph contractible to H has all its vertices the same colour.

Proof. We may assume that G is connected. Choose $v \in V(G)$ and for each $i \geqslant 0$ let V_i be the set of vertices of G at distance i from v. (So $V_0 = \{v\}$ and $V_1 = N(v)$.) Colour the vertices of V_i blue if i is odd and red if i is even. No induced subgraph $\langle V_i \rangle$ can have H as a minor since, if it did, then G would have a $K_1 + H$ minor (contract all the edges in $\langle \{v\} \cup V_1 \cup \ldots \cup V_{i-1} \rangle$), $\Rightarrow\Leftarrow$. Since H is connected, this colouring will do. //

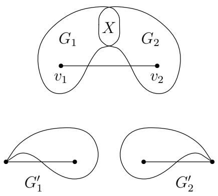
Corollary 12.1. Hadwiger's conjecture holds for $r = 2$, since every graph in $\mathcal{G}(K_3) = \mathcal{G}(K_1 + K_2)$ can be 2-coloured with no monochromatic K_2. //

Corollary 12.2. Every outerplanar graph is $(2, 2)^*$-colourable, since $\{\text{outerplanar graphs}\} \subset \mathcal{G}(K_{2,3}) \subset \mathcal{G}(K_1 + K_{1,3})$, and all such graphs can be 2-coloured with no monochromatic $K_{1,3}$. //

Corollary 12.3. Every $K_{2,s}$-minor-free graph is $(2, s-1)^*$-colourable $(s \geqslant 1)$, since $\mathcal{G}(K_{2,s}) \subset \mathcal{G}(K_1 + K_{1,s})$, and all such graphs can be 2-coloured with no monochromatic $K_{1,s}$. //

Theorem 13. (D. R. Woodall, 1990.) Suppose H is connected, $H \neq K_1$, and $G \in \mathcal{G}(K_2 + H)$. Then G can be (improperly) 3-coloured with no monochromatic H minor.

Proof. We prove by induction that this can be done so that any two specified adjacent vertices v_1, v_2 are properly coloured (i.e., coloured differently from their neighbours). It suffices to prove this when G is connected and v_1v_2 is not a cut-edge. Let $X \neq \emptyset$ be a minimal set of vertices separating v_1 from v_2 in $G - v_1v_2$. Then G has $K_2 + \langle X \rangle$ as a minor, and so $\langle X \rangle$ is H-minor-free. Let



$G - v_1v_2 = G_1 \cup G_2$ where G_1 and G_2 are connected, $G_1 \cap G_2 = \langle X \rangle$, $v_1 \in G_1$ and $v_2 \in G_2$. For $i \in \{1, 2\}$, form G_i' from G by contracting all edges in G_i, so that all vertices of X are identified with v_i. Then $G_1', G_2' \in \mathcal{G}(K_2 + H)$ and so by induction we can 3-colour them with no monochromatic H minor and with v_1, v_2 properly coloured. Do this so that

in G_1', v_1 is green and v_2 is blue,
in G_2', v_1 is red and v_2 is green.

This gives a 3-colouring of G as required, with v_1 red, v_2 blue and all vertices of X green. //

Corollary 13.1. Hadwiger's conjecture holds for $r = 3$, since every graph in $\mathcal{G}(K_4) = \mathcal{G}(K_2 + K_2)$ can be 3-coloured with no monochromatic K_2. //

Corollary 13.2. Every planar graph is $(3, 2)^*$-colourable, since $\{\text{planar graphs}\} \subset \mathcal{G}(K_{3,3}) \subset \mathcal{G}(K_2 + K_{1,3})$, and all such graphs can be 3-coloured with no monochromatic $K_{1,3}$. //

Corollary 13.3. Every $K_{3,s}$-minor-free graph is $(3, s-1)^*$-colourable ($s \geqslant 1$), since $\mathcal{G}(K_{3,s}) \subset \mathcal{G}(K_2 + K_{1,s})$. //

Corollary 13.4. Every $K_{2,s}$-minor-free graph is $(3, s-2)^*$-colourable ($s \geqslant 2$), since $\mathcal{G}(K_{2,s}) \subset \mathcal{G}(K_2 + K_{1,s-1})$. //

Conjecture 6A. Suppose H is connected, $H \neq K_1$, and $G \in \mathcal{G}(K_{r-1} + H)$ where $r \geqslant 2$. Then G can be (improperly) r-coloured with no monochromatic H minor.

Conjecture 6B. Every $K_{r,s}$-minor-free graph is $(r+s-1-d, d)^*$-colourable whenever $0 \leqslant d \leqslant s-1$; in particular, it is (properly) $(r+s-1)$-colourable.

We have proved Conjecture 6A when $r \leqslant 3$ and Conjecture 6B when $r \geqslant 2$ and $r+s-1-d \leqslant 3$. (See Theorem 16 for the case $r = 1$.) We now look at $\mathcal{G}(K_{3,3})$.

K. Wagner (1937) proved that a connected $K_{3,3}$-minor-free graph is either planar or K_5 or has a separating set X comprising one vertex or two adjacent vertices. If $Y \subseteq V(G)$, call a colouring of $V(G)$ *Y-proper* if no vertex of Y has the same colour as any of its neighbours outside Y; two adjacent vertices of Y *may* have the same colour.

Theorem 14. Every $K_{3,3}$-minor-free graph is 5-colourable and $(4, 1)^*$-colourable.

Proof. Let $(k, d) := (5, 0)$ or $(4, 1)$. By induction on $|V(G)|$, we prove the stronger result that if Y comprises one vertex or two adjacent vertices of a $K_{3,3}$-minor-free graph G, and the vertices of Y are coloured (with different colours, if $|Y| = 2$ and $d = 0$), then this colouring of Y can be extended to a Y-proper $(k, d)^*$-colouring of G. This holds by the 4CT if G is planar, and it is easy to see if $G \cong K_5$. It holds by induction if G is not connected. So suppose G has a separating set X comprising one vertex or two adjacent vertices; say $G = G_1 \cup G_2$ where $V(G_1 \cap G_2) = X$, $|V(G_1)| > |X|$, $|V(G_2)| > |X|$ and $Y \subseteq V(G_1)$. By the induction hypothesis, the colouring of Y can be extended to a Y-proper $(k, d)^*$-colouring of G_1. This determines a colouring of X, which can likewise be extended to an X-proper $(k, d)^*$-colouring of G_2. These two colourings together give a Y-proper $(k, d)^*$-colouring of G. The result follows. //

By Corollary 13.2, every $K_{3,3}$-minor-free graph is $(3, 2)^*$-colourable. But for every $d \geqslant 0$, $K_1 + (d+1)K_{1,d+1}$ is a planar

graph that is not $(2, d)^*$-colourable.

7. Improper list colourings.

An *$(L, d)^*$-colouring* is an L-colouring in which no vertex has more than d neighbours with the same colour as itself. A graph G is *k-choosable with defect d*, or *$(k, d)^*$-choosable*, if it has an $(L, d)^*$-colouring whenever $|L(v)| \geqslant k$ $\forall\, v \in V(G)$. So $(k, 0)^*$-choosable = k-choosable.

Theorem 15. (D. R. Woodall, 2000.) Suppose H is connected, $H \neq K_1$, and $G \in \mathcal{G}(K_1 + H)$. Suppose each vertex v is given a list $L(v)$ of (at least) 2 colours. Then G has an L-colouring with no monochromatic H minor.

Proof. We prove the following stronger result: if $z \in V(G)$ and $|L(v)| \geqslant 1$ if $v \in N(z)$ and $|L(v)| \geqslant 2$ if $v \in V(G) \setminus (N(z) \cup \{z\})$, then $G - z$ has an L-colouring with no monochromatic H minor. This will prove the theorem, since we can first give z an arbitrary colour c_z from its list, remove c_z from the lists of all z's neighbours, and then complete the colouring by the stronger result.

Proof of the stronger result by induction on $|V(G)|$. Let c be a colour in the list of some vertex in $N_G(z)$, and let $Z_c := \{v \in N_G(z) : c \in L(v)\}$. Give colour c to all vertices in Z_c. Form (G', L') from (G, L) by contracting all edges vz s.t. $v \in Z_c$ and then removing c from the list of every vertex in $N_{G'}(z)$; then $|L'(v)| \geqslant 1$, $\forall\, v \in N_{G'}(z)$. Since G is $(K_1 + H)$-minor-free, $\langle Z_c \rangle$ is H-minor-free and G' is $(K_1 + H)$-minor-free. By induction, there is an L'-colouring of $G' - z$ with no monochromatic H minor, and this gives the required L-colouring of G. //

Corollary 15.1. Every K_3-minor-free graph is 2-choosable, since $\mathcal{G}(K_3) = \mathcal{G}(K_1 + K_2)$. //

Corollary 15.2. Every outerplanar graph is $(2, 2)^*$-choosable, since $\{\text{outerplanar graphs}\} \subset \mathcal{G}(K_{2,3}) \subset \mathcal{G}(K_1 + K_{1,3})$. //

Corollary 15.3. Every $K_{2,s}$-minor-free graph is $(2, s-1)^*$-choosable ($s \geqslant 1$), since $\mathcal{G}(K_{2,s}) \subset \mathcal{G}(K_1 + K_{1,s})$. //

Conjecture 7A. Suppose H is connected, $H \neq K_1$, and $G \in \mathcal{G}(K_2 + H)$. Suppose each vertex v is given a list $L(v)$ of (at least) 3 colours. Then G has an L-colouring with no monochromatic H minor.

Conjecture 7B. Every $K_{r,s}$-minor-free graph is $(r+s-1-d, d)^*$-choosable whenever $0 \leqslant d \leqslant s-1$; in particular, it is (properly) $(r+s-1)$-choosable.

Conjecture 7B is known to be true when $r \leqslant 2$. Here is the case $r = 1$.

Theorem 16. Let G be a $K_{1,s}$-minor-free graph (which implies $\Delta(G) \leqslant s-1$).
(a) G is $(k, d)^*$-choosable whenever $k(d+1) \geqslant s$.
(b) G is $(s-d, d)^*$-choosable whenever $0 \leqslant d \leqslant s-1$.

Proof. (b) follows from (a), since if $0 \leqslant d \leqslant s-1$ and $k = s-d$ then $k(d+1) = (s-d)(d+1) = s + d(s-1-d) \geqslant s$.

To prove (a), suppose $|L(v)| \geqslant k$, $\forall\, v \in V(G)$. Let λ be an L-colouring of G with as few bad edges (i.e., edges uv with $\lambda(u) = \lambda(v)$) as possible. Suppose some vertex v has $\geqslant d+1$ neighbours coloured $\lambda(v)$. Since $d(v) \leqslant s-1 < k(d+1)$, some colour c in $L(v)$ occurs on at most d neighbours of v. Changing $\lambda(v)$ to c would reduce the number of bad edges, $\Rightarrow\Leftarrow$. So λ is an $(L, d)^*$-colouring. //