G12RAN BLOW BY BLOW 2001-2002

Lecture 1: *Chapter 1. Properties of the real numbers* **I. Review of notation, definitions and results from earlier modules** The sets of natural numbers, integers, rational numbers. The irrational numbers. Other subsets of $\mathbb{R}$ including intervals. Convergence of sequences.

Lecture 2: Divergence of sequences. Further revision of results about sequences including the Algebra of Limits, Monotone Sequence Theorem and Sandwich Theorem. The completeness of $\mathbb{R}$ (existence of supremum/infimum for non-empty, bounded sets). Density of the rationals and the irrationals in $\mathbb{R}$: (as in G1ALIM) every open interval (a, b) in $\mathbb{R}$ contains infinitely many rational numbers and infinitely many irrational numbers.

Lecture 3: **II. Further properties** The nested intervals theorem: Given non-empty closed intervals $[a_n, b_n]$ such that, for all $n \in \mathbb{N}$, $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$, then there must be at least one point c common to all of these closed intervals. If the lengths of the intervals involved tend to 0, then there is *exactly* one such point c. Failure of the nested intervals theorem for open intervals (exercise in first problem class). Failure of the monotone sequence theorem and the nested intervals theorem for $\mathbb{Q}$ (exercise): $\mathbb{Q}$ is incomplete. Intersections of infinitely many sets: notation and examples.

Lecture 4: *Chapter 2. Functions and sets:* The domain and codomain of a function. Cartesian products. Graphs of functions. Examples of functions between subsets of $\mathbb{R}$ including characteristic functions. Injections (injective/1-1 functions), surjections (surjective/onto functions) and bijections (bijective functions) revised. An example of a bijection between $\mathbb{N}$ and $\mathbb{Z}$: $\mathbb{Z}$ is countable. Working definition for first problem class: a set is countable if it is empty or else there is a sequence of elements which includes (uses up) all of the elements of the set.

Lecture 5: More revision of injections and surjections. Bijections. Inverse functions for bijections. Addition and multiplication for real-valued functions (pointwise operations). Composition of functions ($f \circ g$ or $f(g)$). Using our working definition of countability: the empty set $\emptyset$ is countable; every finite set is countable; $\mathbb{Z}$ is countable; $\mathbb{N} \times \mathbb{N}$ is countable. (Unions of two countable sets and countability of $\mathbb{Q}$ were covered in the first problem class). Uncountable sets include $\mathbb{R}$, $[0, 1]$, $[0, 1)$ (proofs next time).

Lecture 6: Uncountability of $[0, 1)$ (Cantor diagonalization argument). Uncountability of the set of all subsets of $\mathbb{N}$ ($2^{\mathbb{N}}$, the power set of $\mathbb{N}$). Two sets have the same cardinality if there is a bijection between them. Comparison of our working definition of countable with the standard definition (countable sets are sets which are finite or have the same cardinality as $\mathbb{N}$; all other sets are uncountable) using the correspondence between functions from $\mathbb{N}$ to a set X and sequences of elements of X.

Lecture 7: Other standard results about countability: images under functions of countable sets are countable (i.e. if X is a countable set, Y is a set and f is a surjection from X to Y then Y must also be countable, proof left as an exercise); if there is an injection f from a set Y into a countable set X then Y must also be countable; in particular, subsets of countable sets are countable; Since $[0, 1)$ is uncountable, so is $\mathbb{R}$. Since $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$ it follows that $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. The product of two countable sets and countable unions of countable sets were done as the first homework assignment.

Chapter 3. Limit values for functions: Introductory examples illustrating the two types of one-sided limits for functions: $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$. These one-sided limits may or may not exist. When both exist, they may be different. The value of the function at the point a is irrelevant here, and need not even be defined. We will work with definitions in terms of sequences rather than the more standard $\epsilon - \delta$ definitions.

Lecture 8: One-sided limits (as above) and two-sided limits ($\lim_{x\to a} f(x)$): definitions in terms of sequences. Punctured neighbourhoods. Examples where one-sided limits exist and are the same, where they exist and are different and where they do not exist. Connection with existence of two-sided limits.

Lecture 9: Definitions in terms of sequences and examples of some further concepts: divergence of functions to $\pm\infty$ as x approaches a in the various possible ways; convergence of $f(x)$ or divergence of $f(x)$ to $\pm\infty$ as $x \to \pm\infty$. The substitution $t = 1/x$, and investigation of the behaviour of the function $\cos(1/x)$. Brief discussion of the four kinds of monotone functions: more details in the printed notes.

Lecture 10: *Chapter 4. Sequences and continuous functions:* Continuity and discontinuity of real-valued functions defined on intervals in terms of one/two sided limits and in terms of limits of sequences. Discussion of the different ways in which a function can fail to be continuous at a point. Standard functions are continuous where they are defined. Sequences of points in the interval I which converge to some point outside I are not important when checking continuity.

Lecture 11: Subsequences of sequences. The Bolzano-Weierstrass theorem: every bounded sequence of real numbers has at least one convergent subsequence.

Lecture 12: New continuous functions from old: sums, products and quotients of continuous functions. Composition of continuous functions. Brief discussion of the boundedness theorem and the intermediate value theorem.

Lecture 13: Further discussions of the boundedness theorem and the intermediate value theorem. Proof of the boundedness theorem.

Lecture 14: Motivation: why do we need to prove intuitively obvious facts? Proof of the intermediate value theorem. The continuous image of an interval is an interval. The continuous image of a closed and bounded interval is a closed and bounded interval.

Chapter 5. Differentiability: Differentiability at a point, interpretation in terms of limiting gradients.

Lecture 15: Functions which are differentiable on intervals, including closed intervals $[a, b]$ (one-sided derivatives defined at the endpoints) and $\mathbb{R}$ (n.b. $\mathbb{R}$ is an interval). Differentiable functions must be continuous, but continuous functions need not be differentiable. In fact there are functions which are continuous everywhere but differentiable nowhere. All the usual standard functions (constant functions, polynomials, rational functions, trigonometric functions etc.) are differentiable where they are defined. (See books for details: standard facts about these functions may be assumed.) The algebra of derivatives: sums, products and quotients of differentiable functions are differentiable (avoiding division by 0). Proof of product rule. Proofs of sum rule and quotient rule left as an exercise. The chain rule for differentiation: standard false proof given. Exercise: find the mistake!

Lecture 16: False proof of chain rule is valid provided that $f(x) \neq f(a)$ for all x which are 'near enough to a and not equal to a' (i.e. on some punctured neighbourhood of a). Problems with functions that cross horizontal lines too often, e.g. $f(x) = x^3 \sin(1/x)$. Chord functions. Correct proof of the chain rule. Statements of Rolle's Theorem and the Mean Value Theorem, with diagrams (proofs deferred). Reminder: how to find the greatest and least values of a function on a closed interval (example in problem class). The implications of the MVT: connections with monotonicity. For example, proof given that if $f'(x) > 0$ for all x in (a, b) then f is strictly increasing on (a, b) (four other similar results use the other inequality/equality signs). Brief discussion of Lipschitz continuity. Boundedness of f' on an interval implies that f is Lipschitz continuous there (proof deferred).

Lecture 17: More details on Lipschitz continuity. Proofs of Rolle's Theorem and MVT.

Lecture 18: Higher order derivatives: n times differentiable functions, n times continuously differentiable functions. Infinitely differentiable functions. Power series for $\exp(x)$, $\sin(x)$ and $\cos(x)$ (angles must be measured in radians): forward look to Taylor's Theorem. *Chapter 6. L'Hôpital's rule and Taylor's theorem:* Limits of quotients of functions $f(x)/g(x)$. When possible, use the algebra of limits! Indeterminate forms of type '0/0' and of type '∞/∞'. Various forms of L'Hôpital's rule stated: one-sided or two-sided limits, or limits as $x \to +\infty$ or $-\infty$. Examples of applications of L'Hôpital's rule.

Lect 19-20: Revised statement of L'Hôpital's rule (in terms of generalised notion of limit, 'lim', which includes possible divergence of the function to ∞). Proof of one easy version of L'Hôpital's rule. Other versions have proofs on handout. Students should know statements of all versions of L'Hôpital's rule and be able to use them, but the only version whose proof is examinable is the one proved in lectures. Taylor series and Maclaurin series. Examples of functions where the Taylor (or Maclaurin) series gives: (i) the original function, as hoped (e.g. $\sin(x)$, $\cos(x)$, polynomials) (ii) the constant function 0 (e.g. the infinitely differentiable function mentioned above which is not constant but all of whose derivatives at 0 are 0). Taylor's theorem and its application to error estimates (e.g. estimation of $\cos(0.1)$ with error less than 10^{-5}). The proof of Taylor's theorem is not examinable, but the statement is, as are its applications. The generalized second derivative test.

$\epsilon - \delta$ revisited and Uniform Continuity Short additional section, including the definition of uniform continuity and the connections between continuity, uniform continuity and Lipschitz continuity.

Lecture 21: Student opinion forms issued. Proof of the fact that every continuous function on a closed and bounded interval is uniformly continuous.

Chapter 7. Integration: Brief discussion of antiderivatives (also called primitives) and the informal notion of area under the curve. The main result of this section is the Fundamental Theorem of Calculus which implies that every continuous function has an antiderivative. Area under the curve does not appear to make much sense for the characteristic function of $\mathbb{Q}$. Brief discussion of the idea behind Riemann integration: using rectangles to estimate areas from above and below.

Lecture 22: *The statements and applications of results in this section are all examinable but the only proofs which are examinable are the ones given in lectures.* Partitions of intervals. Riemann upper and lower sums for a bounded function f corresponding to a given partition P. Refinements of partitions. Each Riemann lower sum $L(P, f)$ is less than or equal to every Riemann upper sum $U(Q, f)$. The Riemann upper and lower integrals. Riemann integrable functions (where the lower and upper Riemann integrals are the same) and the Riemann integral of such functions (equal to both the Riemann upper and lower integrals in this case). Riemann's criterion for integrability. Continuous real-valued functions on closed and bounded intervals are Riemann integrable. Other elementary facts about Riemann integration. Statement and brief discussion (but not proof) of the (first) fundamental theorem of calculus: if f is continuous on $[a, b]$, define $F(x)$ by

$$F(x) = \int_a^x f(t)\ \mathrm{d}t.$$

Then $F'(x) = f(x)$ for all $x \in (a, b)$. (For the proof, which is fairly straightforward, see, for example, Dr Langley's notes.) Thus continuous functions on intervals always have antiderivatives. Improper Riemann integrals: see final question sheet.