G12RAN: Real Analysis

2. Functions and sets

I. Functions

Let A, B be sets (often, in this module, subsets of $\mathbb{R}$). A function $f : A \to B$ is some rule that assigns to each element of A a *unique* element of B, denoted by $f(x)$. The set A is called the *domain* of f and the set B is called the *codomain* of f.

You have met, and should be familiar with, many standard examples of functions in earlier modules, usually real-valued (i.e. with codomain $\mathbb{R}$). In particular you should be familiar with the nature of trigonometric functions, exponential and logarithmic functions.

You should also know about polynomials and rational functions: polynomial functions of x are functions of the form $a_0 + a_1x + \ldots + a_nx^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants (real numbers), and rational functions are functions of the form $p(x)/q(x)$ where p and q are polynomials. Polynomials are defined for all values of x, while rational functions are defined where the denominator is non-zero.

You may be less familiar with *characteristic functions* (also called *indicator functions*).

Definition Let E be a subset of $\mathbb{R}$. The characteristic function of E, χ_E is defined by

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E; \\ 0 & \text{otherwise.} \end{cases}$$

Cartesian products and graphs of functions

Definition Given two sets A, B, the *Cartesian product*, $A \times B$, is defined to be the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$.

For example, if $A = B = \mathbb{R}$ then $A \times B = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ which is the usual set of all points in two-dimensional space, $\mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$.

Definition Let A, B be sets and let f be a function from A to B. Then the *graph* of f is the set $\{(x, f(x)) : x \in A\}$.

Note that the graph of f is a subset of $A \times B$. If f is a function from $\mathbb{R}$ to $\mathbb{R}$, then the graph of f is (as you would expect) a subset of $\mathbb{R}^2$.

Bijections and inverse functions

Definition Let A, B be sets and let f be a function from A to B. We say that f is *injective* (or *one to one*, or $1-1$) if, whenever x, y are elements of A with $x \neq y$ then we have $f(x) \neq f(y)$. (Or, equivalently, $f(x) = f(y) \Rightarrow x = y$; no value is taken more than once.) In this case we say that f is an *injection*.

The function f is *surjective* (or *onto*) if for every $b \in B$ there is *at least one* $a \in A$ with $f(a) = b$. (Every value in B is taken at least once.) In this case we say that f is a *surjection.*

The function f is *bijective* if it is both injective and surjective. We call such a function f a *bijection.* This means that for all $b \in B$ there is *exactly* one element a of A such that $f(a) = b$. In this case, for such a, b, we define $f^{-1}(b) = a$. This gives us a function $f^{-1} : B \to A$ with the property that $f(f^{-1}(b)) = b$ for all $b \in B$ and also $f^{-1}(f(a)) = a$ for all $a \in A$. The function f^{-1} is called the *inverse function* of f. It is clear that f^{-1} is a bijection from B to A, and that the inverse function of f^{-1} is f.

New functions from old!

Given a set A and two functions f, g from A to $\mathbb{R}$, we can define two new functions, $f+g$ the *pointwise sum* of f and g, and fg, the *pointwise product* of f and g in a natural way. For $a \in A$ we set $(f + g)(a) = f(a) + g(a)$ and $(fg)(a) = f(a)g(a)$.

If instead we have sets X, Y, Z and functions $f : X \to Y$ and $g : Y \to Z$ then we can define the *composite function* $g \circ f : X \to Z$ as follows: for $x \in X$ we set $(g \circ f)(x) = g(f(x))$. (Some people use the notation $g(f)$ instead of $g \circ f$.)

Exercise Let X, Y, Z, f and g be as above and suppose that f and g are both bijections. Show that $g \circ f$ is also a bijection, and that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Countable sets and uncountable sets

Definition We say that two sets X, Y *have the same cardinality* (or *the same power*) if there exists a bijection from X to Y.

In the case of finite sets (sets with only finitely many elements) it is clear that two sets have the same cardinality if and only if they have the same number of elements. In particular, we count $\emptyset$ as a finite set: it is the only set with this cardinality (no elements).

Definition A set X is *countable* if it is either finite or has the same cardinality as $\mathbb{N}$.

Warning: some authors do not regard finite sets as countable. To avoid ambiguity, we shall often refer to countable sets which are not finite as *countably infinite* sets (also known as *denumerable* sets).

If an (infinite) set X is not countable then it is *uncountable.*

There is a close connection between functions from $\mathbb{N}$ to X and sequences of elements of X (in fact there is essentially no difference). This leads to the following.

Proposition 2.1 Let X be a set. Then X is countably infinite if and only if there is a sequence $(x_n) \subseteq X$ such that every element of X appears exactly once in the sequence (x_n).

Here are some of the results that we shall meet on countability and uncountability.

Proposition 2.2 The set $\mathbb{Z}$ is countable.

Proposition 2.3 The interval $[0, 1)$ is uncountable.

Similarly $(0, 1)$ is uncountable and $\mathbb{R}$ is uncountable.

Lemma 2.4 Let X be a set and suppose that there is a surjection f from $\mathbb{N}$ onto X. Then X is countable.

Lemma 2.5 Let X be a set. Then X is countable if and only if *either* X is empty *or* there is a sequence $(x_n) \subseteq X$ such that every element of X appears *at least* once in the sequence (x_n).

Note that we now allow repeats in the sequence, which allows us to deal with finite sets.

Exercise Let X be any countable set. Prove the following statements.

(i) Let Y be a set and suppose that there is a surjection f from X onto Y. Then Y is countable.

(ii) Let Y be a set and suppose that there is an injective function f from Y to X. then Y is countable.

(iii) Every subset of X is also countable.

Proposition 2.6 The set $\mathbb{N} \times \mathbb{N}$ is countable.

This is in fact a special case of part of the following proposition, which is an exercise on the first question sheet.

Proposition 2.7

(i) Let A, B be countable sets. Then $A \cup B$ and $A \times B$ are both countable.

(ii) A countable union of countable sets is countable: let $A_1, A_2, A_3, \ldots$ be countable sets. Then the set

$$\bigcup_{n \in \mathbb{N}} A_n$$

is also countable.

We shall also see (on the first question sheet) that $\mathbb{Q}$ is countable. As a consequence of this and Proposition 2.7 we can see that $\mathbb{R} \setminus \mathbb{Q}$ is uncountable. In other words, although both the rational numbers and the irrational numbers are dense in $\mathbb{R}$, there are 'more' irrational numbers than there are rational numbers!