3. Limit values for functions

Punctured neighbourhoods

Let $a \in \mathbb{R}$. Then a *punctured neighbourhood* of a is a set of the form

$$A = (b, a) \cup (a, c)$$

for some $b < a$ and some $c > a$. (Note that $A = (b, c) \setminus \{a\}$, and in particular that a is NOT an element of A.)

Many of the concepts in this module have several equivalent definitions. We will work mainly with sequences, and these are the definitions given first below.

You should be warned that most text books give definitions in terms of ϵ and δ. We will not work much with these equivalent definitions, but they are included below for information.

Sequence definitions

Definition Let $a \in \mathbb{R}$ and let f be a real-valued function defined on some punctured neighbourhood A of a. Let $L \in \mathbb{R}$. Then we say that *the limit of $f(x)$ as x tends to a exists and equals L* if, for every sequence (x_n) in $A \setminus \{a\}$ which converges to a we have

$$\lim_{n \to \infty} f(x_n) = L.$$

NOTE that the value of $f(a)$ (even if this is defined) is NOT relevant when investigating this limit: you only need to look at values of x which are near to a but not equal to it. So the sequences (x_n) never include a as one of the terms.

If the limit exists as above, we also write

$$\lim_{x \to a} f(x) = L$$

or

$$f(x) \to L \text{ as } x \to a.$$

We also have one-sided versions of limits (with similar alternative notation available).

Definition Let a, b in $\mathbb{R}$ with $a < b$ and suppose that f is a real-valued function defined on (a, b). Let $L \in \mathbb{R}$. Then we say that

$$\lim_{x \to a+} f(x) = L$$

if, for every sequence (x_n) in (a, b) which converges to a we have

$$\lim_{n \to \infty} f(x_n) = L.$$

Similarly, we say that

$$\lim_{x \to b-} f(x) = L$$

if, for every sequence (x_n) in (a, b) which converges to b we have

$$\lim_{n \to \infty} f(x_n) = L.$$

It is easy to see for a real-valued function f defined on a punctured neighbourhood A of a point a that $\lim_{x \to a} f(x) = L$ if and only if both $\lim_{x \to a-} f(x) = L$ and $\lim_{x \to a+} f(x) = L$.

Exercise. In terms of one-sided limits, what are the different ways in which a function f can fail to have a two-sided limit at a?

Similar concepts, including divergence of $f(x)$ to $+\infty$ or $-\infty$ and limits as x tends to $+\infty$ or $-\infty$ will be discussed in lectures.

ϵ-δ definitions

These are the standard definitions given in most textbooks. We will not use them often in this module, but they are included here for information.

Definition Let $a \in \mathbb{R}$ and let f be a real-valued function defined on some punctured neighbourhood A of a. Let $L \in \mathbb{R}$. Then we say that *the limit of $f(x)$ as x tends to a exists and equals L* if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that, for all $x \in \mathbb{R}$ with $0 < |x - a| < \delta$, we have $|f(x) - L| < \epsilon$. (Or, in other words, each of the intervals $(a - \delta, a)$ and $(a, a + \delta)$ is mapped by f into some subset of the open interval $(L - \epsilon, L + \epsilon)$.)

The smaller ϵ is, the smaller a value of δ we are likely to need to ensure that the condition above holds.

There are similar definitions for one-sided limits and for limits as $x \to +\infty$ or $-\infty$, *etc.* of $f(x)$.

Examples

Typical examples of limits are

$$\lim_{x\to 2}(x^2 + 1) = 5.$$

while neither of the limits

$$\lim_{x\to 0}\frac{x}{|x|}, \quad \lim_{x\to 3}\chi_{\mathbb{Q}}(x)$$

exist.

Monotone functions

Let f be a real-valued function defined on an interval I. We say that f is *increasing* (or *nondecreasing*) on I if whenever x, y are in I with $x \le y$ then we have $f(x) \le f(y)$. We say that f is *strictly increasing* on I if whenever x, y are in I with $x < y$ then we have $f(x) < f(y)$. Thus a strictly increasing function is precisely the same as a nondecreasing function which is also $1 - 1$.

Decreasing (or *nonincreasing*) functions and *strictly decreasing functions* are defined similarly.

A monotone function (on I) is a function which is either increasing or decreasing (or both: constant functions are allowed).

Results about monotone functions are similar to the monotone sequence theorem. For example, if f is a monotone increasing function on an interval (a, b) and f is bounded above, then $f(x)$ must converge to a limit as $x \to b-$.

Exercise. Show that, with a, b, f as above, we must have

$$\lim_{x\to b-} f(x) = \sup\{f(t) : t \in (a, b)\}.$$