G12RAN: Real Analysis
7. Integration

You will probably be familiar with integration as a form of anti-differentiation, and as 'area under the curve'. The problem is: how do you know which functions have anti-derivatives? (An *anti-derivative* for a function f is another function F which satisfies $F' = f$. Anti-derivatives are also called *primitives*). On question sheet 5 there is an example of a function which has no anti-derivative (the characteristic function of a set with just one point). The main result of this chapter is the (first) Fundamental Theorem of Calculus, one implication of which is that *every continuous, real-valued function on an interval has an anti-derivative.* Of course, some discontinuous functions also have anti-derivatives (can you think of an example?).

For a bounded, real-valued function f defined on an interval $[a, b]$, we will discuss *partitions* P of $[a, b]$ and the corresponding *Riemann upper sum* and *Riemann lower sum* for f on $[a, b]$ (denoted by $U(P, f)$ and $L(P, f)$ respectively).

Roughly speaking, $U(P, f)$ and $L(P, f)$ represent the integrals of certain easy to consider 'staircase' functions (functions which are constant on the open intervals between vertices of the partition), one staircase function being as small as possible without falling below f, the other being as large as possible without climbing above f. It is not hard to prove the following fact:

Lemma *Let P and Q be partitions of $[a, b]$. Then $L(P, f) \leq U(Q, f)$ (i.e. every Riemann lower sum is no greater than any other Riemann upper sum).*

This allows us to define the Riemann lower and upper integrals for f over the interval $[a, b]$ by, respectively,

$$\underline{\int_a^b} f(x)\ \mathrm{d}x = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}$$

$$\overline{\int_a^b} f(x)\ \mathrm{d}x = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}.$$

We say that the function f is *Riemann integrable* over $[a, b]$ if

$$\underline{\int_a^b} f(x)\ \mathrm{d}x = \overline{\int_a^b} f(x)\ \mathrm{d}x$$

(i.e. if the lower integral is equal to the upper integral). *In this case* we define the Riemann integral of f from a to b to be the common value:

$$\int_a^b f(x)\ \mathrm{d}x = \underline{\int_a^b} f(x)\ \mathrm{d}x = \overline{\int_a^b} f(x)\ \mathrm{d}x.$$

Unbounded functions on an interval $[a, b]$ are declared *not* to be (proper) Riemann integrable (it is hard to define lower and upper sums for such functions). However they may be 'improper' Riemann integrable (see question sheet 5).

Note that the Riemann lower integral is always less than or equal to the Riemann upper integral, every Riemann lower sum is less than or equal to the lower integral, and every Riemann upper sum is greater than or equal to the upper integral.

It is easy to show that constant functions are Riemann integrable (with the obvious integral) but that functions like $\chi_{\mathbb{Q}}$ are not (see question sheet 5).

Theorem (Riemann's criterion for integrability) *Let f be a bounded, real-valued function on an interval $[a,b]$. Then T.F.A.E.*
(i) f is Riemann integrable on $[a,b]$,
(ii) For all $\epsilon > 0$ there is a partition P_ϵ of $[a,b]$ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.

From this, and the fact that every continuous, real-valued function on $[a,b]$ must also be uniformly continuous, we deduce easily:

Theorem *Every continuous, real-valued function on an interval $[a,b]$ is Riemann integrable on $[a,b]$.*

Other examples of functions which are always Riemann integrable on intervals $[a,b]$ include all monotonic functions (exercise!).

It is not too hard (but we will not have time to prove this in this module) to show that if f and g are Riemann integrable on an interval $[a,b]$, then so are the functions $f+g$, fg and $|f|$ (here $|f|(x)$ is defined to be $|f(x)|$, of course).

We now come to the final (and main) result of this chapter.

Theorem (First Fundamental Theorem of Calculus) *Let f be a continuous, real-valued function on $[a,b]$. For $x \in [a,b]$ define*

$$F(x) = \int_a^x f(t)\ \mathrm{d}t.$$

Then F is continuous on $[a,b]$, and is differentiable on (a,b), with $F'(x) = f(x)$.

From this it follows easily that continuous, real-valued functions on intervals always have anti-derivatives. It also shows that anti-differentiation is the correct way to integrate continuous functions.