G12RAN Real Analysis: Exercises 1

You should attempt questions 1 to 3 in the first problem class. If you finish early then you can try questions 4 and 5. Try to solve the questions in advance if possible! At the very least, **make sure that you are familiar with the definitions of the concepts mentioned in these questions**. (Otherwise you will have to spend time looking up the definitions during the class, instead of thinking about the questions.)

Answers to questions 6 and 7 should be handed in at the end of the lecture on Friday 18th October.

Note that quality of exposition is very important: poor exposition/expression in the examination will cost marks. You must always justify your answers.

1. (a) Find an example of a surjection f from $\mathbb{R}$ onto $\mathbb{R}$ which is not an injection.
(b) Find an example of an injection from $\mathbb{R}$ to $\mathbb{R}$ which is not a surjection.
(c) Find a bijection between the two open intervals $(0, 1)$ and $(0, \infty)$.

2. Find an example of a sequence of open intervals (a_n, b_n) with $a_n < b_n$ for all n, and such that $(a_n, b_n) \supseteq (a_{n+1}, b_{n+1})$ for all n, and yet there is no real number c which is in the intersection of all of these open intervals.

3. Let A, B be countable sets. Prove that $A \cup B$ is countable. (You may find it useful to begin by considering the case where at least one of the sets is empty).

4. Prove that the set $\mathbb{Q}$ is countable.

5. Let A, B be non-empty subsets of $\mathbb{R}$ which are bounded above.
(a) Prove that $\sup(A \cup B) = \max\{\sup A, \sup B\}$.
(b) Now suppose that $A \cap B \neq \emptyset$. Is it necessarily the case that $\sup(A \cap B) = \min\{\sup A, \sup B\}$? [Give a proof or a counterexample].

6. Let A, B be countable sets. Prove that $A \times B$ is countable. (You may find it useful to begin by considering the case where at least one of the sets is empty).

7. Let $A_1, A_2, \ldots$ be countable sets. Set $A = \bigcup_{n=1}^{\infty} A_n$ (recall that this means the same thing as $\bigcup_{n\in\mathbb{N}} A_n$). Prove that A is countable (so a countable union of countable sets is countable).

8. Define a function h from $\mathbb{R}$ to $\mathbb{R}$ as follows: let $h(x)$ be 0 if $x \neq 0$ and let $h(0) = 1$ (so in fact h is the characteristic function of the single-point set $\{0\}$). Determine $\lim_{x\to 0} h(x)$ and $\lim_{x\to 0} h(h(x))$. Comment on your answer.

9. Let (a_n), (b_n) be sequences of real numbers and let $c \in \mathbb{R}$. Consider the sequence $a_1, b_1, a_2, b_2, \ldots$. Call this new sequence (c_n) (so $c_{2k} = b_k$ while $c_{2k-1} = a_k$ for each positive integer k).

Prove that the following two statements are equivalent [i.e. (a)$\Rightarrow$ (b) and also (b)$\Rightarrow$(a)].
(a) $\lim_{n\to\infty} a_n = c$ and $\lim_{n\to\infty} b_n = c$;
(b) $\lim_{n\to\infty} c_n = c$.

10. Let f be a real-valued function defined on $(0, \infty)$. Suppose that f satisfies the following condition: for every sequence (x_n) of positive real numbers converging to 0, the sequence $(f(x_n))$ is convergent (to some real number that could possibly depend on the sequence (x_n) chosen). Prove that $\lim_{x\to 0+} f(x)$ exists (i.e. in fact the limit of the sequence $(f(x_n))$ is the same for every such sequence (x_n)). [Hint: use question 9.]