G12RAN Real Analysis

Exercises 1: Solutions to questions 1-5

1 (a) There are many examples. A polynomial which does the trick is

$$f(x) = x^3 - x.$$

This is not injective, since for example

$$f(-1) = f(0) = f(1) = 0.$$

Can you *prove* that f is surjective? For now a sketch of $y = f(x)$ is good enough, but the "real reason" that f is surjective is the Intermediate Value Theorem coming up later in the module.
(An alternative is to use part (c), or even easier, take

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ x+1 & \text{if } x < 0. \end{cases}$$

You should check the properties of this function.)

(b) One standard example is the function

$$f(x) = e^x.$$

As a function from $\mathbb{R}$ to $(0, \infty)$, this is a bijection. But as a function from $\mathbb{R}$ to $\mathbb{R}$ it is injective but not surjective. (You can assume the standard properties of these standard functions.)
An alternative is to use the function f defined by

$$f(x) = \begin{cases} x & (x \geq 0), \\ x-1 & (x < 0). \end{cases}$$

Again, you should check the properties of this function.

(c) The function $f_1(x) = \frac{1}{x}$ is clearly a bijection from $(0, 1)$ to $(1, \infty)$. So the function

$$f_2(x) = \frac{1}{x} - 1 \quad \left[= \frac{1-x}{x}\right]$$

has the required properties.

2 For example, take $a_n = 0$, $b_n = \frac{1}{n}$ $(n \in \mathbb{N})$. Then $(a_n, b_n) = (0, \frac{1}{n})$. These intervals are clearly nested, with $(a_1, b_1) \supseteq (a_2, b_2) \supseteq \dots$. However there is no point of $\mathbb{R}$ common to all these intervals, for if $c > 0$ then $\exists n$ with $\frac{1}{n} < c$ and then $c \notin (0, \frac{1}{n})$. [Any $n > \frac{1}{c}$ will do.] Clearly no c in $(-\infty, 0]$ is in any of the intervals.

In terms of infinite intersections we have

$$\bigcap_{n \in \mathbb{N}} \left(0, \frac{1}{n}\right) = \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset.$$

3 We are given that A, B are countable.

If either A or B is empty then the result is trivial. Otherwise we can find a sequence (a_n) using up all the elements of A (there may be repeats) and a sequence (b_n) using up all the elements of B. But then the sequence $a_1, b_1, a_2, b_2, \ldots$ uses up all the elements of $A \cup B$, and so $A \cup B$ is countable.

[We have made use of the fact that a set X is countable if and only if X is empty or there is a sequence $(x_n)_{n=1}^{\infty}$ which uses up all the elements of X.]

4 There are many ways to do this. One way is to use the fact that $\mathbb{N} \times \mathbb{N}$ is countable, and $\mathbb{Z}$ is countable.

Choose a surjection $f : \mathbb{N} \longrightarrow \mathbb{Z}$, and define $g : \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Q}$ by

$$g((i, j)) = \frac{f(i)}{j} \qquad ((i, j) \in \mathbb{N} \times \mathbb{N}).$$

Then g is a surjection from $\mathbb{N} \times \mathbb{N}$ on $\mathbb{Q}$. Since $\mathbb{N} \times \mathbb{N}$ is countable, so is $\mathbb{Q}$.

Alternatively, consider the following array:

$$\begin{array}{ccccccccccc}
\cdots & \frac{-2}{1} & \longleftarrow & \frac{-1}{1} & & \frac{0}{1} & \longrightarrow & \frac{1}{1} & & \frac{2}{1} & \cdots \\
 & & \searrow & & \nwarrow & & \swarrow & & \nearrow & & \\
\cdots & \frac{-2}{2} & & \frac{-1}{2} & & \frac{0}{2} & & \frac{1}{2} & & \frac{2}{2} & \cdots \\
 & & & & \searrow & & \nearrow & & & & \\
\cdots & \frac{-2}{3} & & \frac{-1}{3} & & \frac{0}{3} & & \frac{1}{3} & & \frac{2}{3} & \cdots
\end{array}$$

every rational number appears (many times) in this array. The path shown shows one of many ways to form a sequence which includes all rational numbers: the sequence starts

$$\frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{-1}{1}, \frac{-2}{1}, \frac{-1}{2}, \frac{0}{3}, \ldots.$$

Note that the second method is similar to the proof that $\mathbb{N} \times \mathbb{N}$ is countable given in lectures.

5 (a) Set $S = \max\{\sup A, \sup B\}$. We show that $S = \sup(A \cup B)$. To prove this we must show (i) S *is* an upper bound for $A \cup B$ and (ii) whenever $t < S$, then t is *not* an upper bound for $A \cup B$.

(i) Let $x \in A \cup B$. Then $x \in A$ or $x \in B$. Suppose that $x \in A$. Then $x \leq \sup A \leq S$, so $x \leq S$. Similarly, if $x \in B$ then $x \leq \sup B \leq S$. Thus in all cases, $x \leq S$ and so S is an upper bound for $A \cup B$.

(ii) Let $t < S$. We show that t is not an upper bound for $A \cup B$. We know $S = \sup A$ or $S = \sup B$. First suppose that $S = \sup A$. Then $t < \sup A$, so, by the definition of sup, there is an x in A with $t < x \leq S$. But then $x \in A \cup B$ and $x > t$, so t is not an upper bound for $A \cup B$. The case where $S = \sup B$ is similar.

(b) No. For a counterexample, consider e.g. $A = \{0, 1\}$, $B = \{0, 2\}$. Then $A \cap B = \{0\}$, so $\sup(A \cap B) = 0$, while $\min(\sup A, \sup B) = \min(1, 2) = 1$.