G12RAN Real Analysis: Exercises 3

You should attempt questions 1 to 3 in the third problem class If you finish early then you can try questions 4 and 5. Try to solve the questions in advance if possible! At the very least, **make sure that you are familiar with the definitions of the concepts mentioned in these questions**.

Answers to questions 6 and 8 should be handed in at the end of the lecture on Friday 15th November.

Note that quality of exposition is very important: poor exposition/expression in the examination will cost marks. You must always justify your answers.

1. The boundedness theorem tells us that every continuous real-valued function on the closed interval $[0, 1]$ is bounded and attains its bounds. For each of the following intervals I give an example of a continuous function f from I to $\mathbb{R}$ which is unbounded, and an example of a function g from I to $\mathbb{R}$ which is continuous and bounded on I, but such that g does not have a maximum value on I.

(i) $I = [0, 1)$; (ii) $I = (1, \infty)$; (iii) $I = [0, \infty)$.

2. Let f, g be continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Suppose that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$, i.e. $f = g$.

3. A function f from $\mathbb{R}$ to $\mathbb{R}$ is said to be *periodic*, and a real number $T > 0$ is said to be a *period* of f if, for all $x \in \mathbb{R}$, $f(x + T) = f(x)$. Suppose that f is a continuous function from $\mathbb{R}$ to $\mathbb{R}$ and that f is periodic. Prove that f is bounded.

4. Which, if any, of the following conditions are satisfied by *all* continuous functions f from $\mathbb{R}$ to $\mathbb{R}$? [Give proofs or counterexamples].

(i) For every bounded sequence of real numbers (x_n), the sequence of function values $(f(x_n))$ is also a bounded sequence.

(ii) For every bounded sequence of real numbers (x_n), the sequence of function values $(f(x_n))$ is convergent.

(iii) There exists a divergent sequence of real numbers (x_n) such that the sequence of function values $(f(x_n))$ is divergent.

5. (i) Prove that the *only* polynomial functions (with real coefficients) p such that, for all $x > 0$, $p(x)/x^2 \in [2, 3]$ are those of the form $p(x) = Ax^2$ for some constant $A \in [2, 3]$.

(ii) Give an example of a continuous function f from $(0, \infty)$ to $\mathbb{R}$ which is not a polynomial but such that, for all $x > 0$,

$$2 \leq \frac{f(x)}{x^2} \leq 3. \qquad (*)$$

(iii) Prove that for every continuous function f from $(0, \infty)$ to $\mathbb{R}$ which satisfies the condition $(*)$ of part (ii) there must be at least one point $c > 0$ such that $f(c) = c$. [Hint: set $g(x) = f(x) - x$ and try to find appropriate endpoints to apply the intermediate value theorem.]

6. Which, if any, of the conditions (i) to (iii) of question 4 are false for *all* discontinuous functions f from $\mathbb{R}$ to $\mathbb{R}$? [Note in each case that it is not helpful to give just one example of a discontinuous function which fails to have the given property.]

7. For $n \in \mathbb{N}$, define a function f_n from $\mathbb{R}$ to $\mathbb{R}$ by $f_n(x) = x^n \sin(1/x)$ when $x \neq 0$, while $f_n(0) = 0$. Prove that f_1 is continuous, but is not differentiable at 0, while f_2 is differentiable, but $(f_2)'$ is not continuous at 0. What can you say about $f_3, f_4, \ldots$?

8. Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that every positive rational number q is a period for f (see question 3 above). Prove that f must be constant.

9. Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Suppose that (i) $f(1) = 1$ and that (ii) $f(x+y) = f(x) + f(y)$ for all real x and y.

(a) Prove that $f(x) = x$ for every positive rational x.

(b) Prove that $f(0) = 0$ and that $f(-x) = -f(x)$ for all real x.

(c) Prove that $f(x) = x$ for all real x.

10. Let $a \in \mathbb{R}$, and, for $x > 0$, define x^a as in the notes by $x^a = \exp(a \log(x))$.

(i) Using the chain rule, and properties of exp and log, verify the standard formula $\frac{\mathrm{d}}{\mathrm{d}x} x^a = a x^{a-1}$.

(ii) Differentiate the functions (defined for $x > 1$) (a) $x^{\sqrt{x}}$ (b) $(\log(x))^{\cos(2x)}$.