Exercises 3: Solutions to questions 1-5

1.(i) Here we can take $f(x) = 1/(1-x)$. This is defined throughout $[0, 1)$ and is continuous there, but is unbounded. We can take $g(x) = x$. This is continuous and bounded on $[0, 1)$ but has no maximum value on I (the supremum of the values taken by g is 1, but g does not take this value on I).

(ii) This time we can take $f(x) = x$. This function is continuous and unbounded on $(0, \infty)$. We can take $g(x) = 1/x$. Clearly g is defined and continuous throughout $(1, \infty)$, and $0 < g(x) < 1$ there. The supremum of the values taken by g is 1, but this value is not achieved.

(iii) We can take $f(x) = x$ again here. The function g is slightly harder, but we can set $g(x) = -1/(1 + x)$. This is defined and continuous throughout the interval $[0, \infty)$, with $-1 \leq g(x) < 0$ for all x in this interval. The supremum of the values taken by g on this interval is 0, but this value is not achieved.

2. Let $x \in \mathbb{R}$. Choose a sequence of rational numbers $(q_n) \subseteq \mathbb{Q}$ which converges to x. Then since f and g are continuous we have

$$f(x) = \lim_{n\to\infty} f(q_n)$$

and

$$g(x) = \lim_{n\to\infty} g(q_n).$$

But f, g agree on the rationals, so $f(q_n) = g(q_n)$ for all n. Taking limits gives $f(x) = g(x)$.

3. This is quite easy given that we know that every continuous function on a closed and bounded interval is bounded (by the boundedness theorem).

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, with some period $T > 0$.

Let us first note that, for all positive integers m, mT is clearly also a period of f. From this we see that for *all* integers k (not necessarily positive) and all $x \in \mathbb{R}$ we have $f(x + kT) = f(x)$.

Next we use the boundedness theorem. Since f is continuous on $[0, T]$, f is bounded there i.e. there are real numbers $c < d$ such that for all $x \in [0, T]$ we have $f(x) \in [c, d]$.

It is now easy to show that, for *all* $x \in \mathbb{R}$, we have $f(x) \in [c, d]$. To see this, take any $x \in \mathbb{R}$. Then there must be an integer n such that $x \in [nT, (n + 1)T]$. We have $x - nT \in [0, T]$ and so $f(x) = f(x - nT) \in [c, d]$ as claimed. Thus the function f is bounded, as required.

4.(i) This condition is true for all continuous functions f from $\mathbb{R}$ to $\mathbb{R}$. Given any such function f and any bounded sequence (x_n) of real numbers, there must be real numbers $a < b$ such that $(x_n) \subseteq [a, b]$. The boundedness theorem tells us that f is bounded on $[a, b]$, and so (since the x_n are in $[a, b]$) the sequence of function values $(f(x_n))$ must be bounded as well.

(ii) There are many continuous functions for which this is false. We need to give a counterexample with brief justification. You can take $f(x) = x$. This continuous function does not satisfy the condition because, for example, the sequence $x_n = (-1)^n$ is bounded, but the sequence of function values $(f(x_n))$ does not converge.

(In fact it is not too hard to show that the only continuous functions satisfying this condition are the constant functions! It is an extra exercise to prove this.)

(iii) Again we provide a counterexample: we can take f to be a constant function, for example $f(x) = 0$. Then the condition fails because it is impossible to find any sequences (x_n) at all (let alone divergent ones) for which the sequence of function values $(f(x_n))$ is divergent. (This sequence of function values is a constant sequence and so converges.)

(In fact, this time the only continuous functions which *fail* the given condition are the constant functions.)

5.(i) This result is intuitively reasonable but the details are a little tricky. Let p be a polynomial satisfying the given conditions. Clearly $p(x)$ is not constant, so we may assume that $p(x)$ is $a_0 + a_1x + \cdots + a_nx^n$ where $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n$ are real constants with $a_n \neq 0$. (Thus p has degree n.) Dividing through by x^n we have (by the algebra of limits)

$$\lim_{x\to+\infty} \frac{p(x)}{x^n} = \lim_{x\to\infty} \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{x^1} + a_n \right) = a_n.$$

We use this to show that $n \leq 2$. First note that we have, for all $x > 0$,

$$x^{n-2}(p(x)/x^n) = p(x)/x^2 \in [2, 3]. \tag{1}$$

Suppose (for contradiction) that $n > 2$. Then $1/x^{n-2} \to 0$ as $x \to +\infty$. Multiplying (1) by $1/x^{n-2}$ gives us (for $x > 0$)

$$2/x^{n-2} \leq \frac{p(x)}{x^n} \leq 3/x^{n-2}.$$

We can now apply the sandwich theorem to see that $\lim_{x\to+\infty} p(x)/x^n = 0$. This contradicts the fact shown above that this limit is a_n, which is not equal to 0. This contradiction shows that $n \leq 2$.

Now we know that $p(x)$ has the form $Ax^2 + Bx + C$. We show that $B = C = 0$ and that $A \in [2, 3]$. First note that $p(x) = x^2(p(x)/x^2)$, so $2x^2 \leq p(x) \leq 3x^2$. By the sandwich theorem we now see that $\lim_{x\to0+} p(x) = 0$. Since this limit is also equal to C, we have shown that $C = 0$ and $p(x) = Ax^2 + Bx$. Now for $x > 0$ we have $p(x)/x = Ax + B$ and this tends to B as $x \to 0+$. However $p(x)/x = x(p(x)/x^2)$ so $2x \leq p(x)/x \leq 3x$ and the sandwich theorem tells us that $\lim_{x\to0+}(p(x)/x) = 0$. Thus $B = 0$. We now have $p(x) = Ax^2$, so $A = p(1) \in [2, 3]$, as required.

(ii) This is quite easy. For example, we may take $f(x) = (5 + \cos(x))x^2/2$. It is clear that this function f has the required properties.

(iii) Let f be any continuous function satisfying condition $(*)$ of part (ii) of this question. Set $g(x) = f(x) - x$. Then $f(1) = f(1)/1^2 \geq 2$, and so $g(1) > 0$. However, $f(1/4) \leq 3/16$ and so $g(1/4) < 0$. Since g is continuous, the intermediate value theorem tells us that there must be a point c between $1/4$ and 1 such that $g(c) = 0$ i.e. such that $f(c) = c$, as required.