G12RAN Real Analysis

Exercises 3: Solutions to questions 6-10

6 The logic of this question is tricky and needs careful thought. For each part of this question we need either to prove that the given condition fails for *all* discontinuous functions from $\mathbb{R}$ to $\mathbb{R}$, or else find a counterexample. Here a counterexample means an example of a *discontinuous* function f from $\mathbb{R}$ to $\mathbb{R}$ which *does* satisfy the given condition. For full marks you need to demonstrate your understanding of this logic by giving specific counterexamples or proofs as appropriate. You must also give some justification for any claims you make.

(i) Here we can find a counterexample. For example, we can take *any* bounded discontinuous function f as a counterexample (there are also some others). For full marks you should give a *specific* counterexample, e.g. $f(x) = \chi_{\mathbb{Q}}(x)$ (the characteristic function of the rationals). Since the function f is bounded, it is clear that the given condition holds. (Full marks for saying this.) In fact, for *every* sequence $(x_n) \subseteq \mathbb{R}$ (not necessarily bounded) we have that $(f(x_n))$ is a bounded sequence.

(ii) This condition fails for all discontinuous functions from $\mathbb{R}$ to $\mathbb{R}$. Let f be any discontinuous function from $\mathbb{R}$ to $\mathbb{R}$. Since discontinuous functions can not be constant, there must be points a, b in $\mathbb{R}$ with $f(a) \neq f(b)$. Now let (x_n) be the sequence $a, b, a, b, a, b, \dots$. This is a bounded sequence of real numbers, but (since $f(a) \neq f(b)$) the sequence $(f(x_n))$ does not converge. Thus the given condition fails for the function f, as required.

(iii) In fact you can prove that this condition only fails for constant functions, so, in fact the condition holds for *all* discontinuous functions. However, we only need to give an example of *one* discontinuous function for which the condition holds. To obtain full marks you need to demonstrate your understanding of the logic of this question by exhibiting one counterexample. For example, we could take $f(x) = \chi_{\mathbb{Q}}(x)$ again (as above). For this function, consider the divergent sequence of real numbers $x_n = (\sqrt{2})^n$. Since the x_n are alternately irrational and rational, we see that $f(x_n)$ is alternately 0 and 1 and so diverges (remember that *diverges* just means 'does not converge '). Thus the condition holds for this choice of function f.

7 A useful fact here is that for a real-valued function h defined on a punctured neighbourhood of the point a,

$$\lim_{x \to a} h(x) = 0 \iff \lim_{x \to a} |h(x)| = 0.$$

[You can check this using sequences, or using the squeeze rule for function limits.]

$$f_1(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Clearly, f_1 is continuous at all points of $\mathbb{R}\backslash\{0\}$. But is f_1 continuous at 0?

We have $0 \leq |f_1(x)| \leq |x|$ for $x \neq 0$, since $\left|\sin(\frac{1}{x})\right| \leq 1$. Thus, by the squeeze rule, $\lim_{x\to 0} |f_1(x)| = 0$, and so $\lim_{x\to 0} f_1(x) = 0$. Since $f_1(0) = 0$, f_1 is continuous. But

$$\frac{f_1(x) - f(0)}{x - 0} = \sin\left(\frac{1}{x}\right),$$

which does not have a limit as $x \to 0$, so f_1 is not differentiable at 0.

$$f_2(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$

Clearly f_2 is differentiable at all points of $\mathbb{R}\backslash\{0\}$, and $f_2'(x) = 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ for $x \neq 0$. It is now clear that $f_2'(x)$ has no limit as $x \to 0$, because if it did then $-\cos\left(\frac{1}{x}\right)$ would have to have the same limit [by the algebra of limits, and noting that $\lim_{x\to 0}\left(-2x\sin\left(\frac{1}{x}\right)\right) = 0$], but this is impossible as $\lim_{x\to 0}\cos\left(\frac{1}{x}\right)$ does not exist.

However, $f_2'(0)$ *does* exist, because, for $x \neq 0$,

$$\frac{f_2(x) - f_2(0)}{x - 0} = x \sin\left(\frac{1}{x}\right) \to 0 \quad \text{as} \quad x \to 0,$$

so $f_2'(0) = 0$. Thus f_2 is differentiable, but is *not* "continuously differentiable".

You can now check that f_3' exists and is continuous, but f_3' is not differentiable at 0 (so f_3 is *not* twice differentiable) but f_3 *is* "continuously differentiable".

Similarly f_4 is twice differentiable, but f_4'' is not continuous, so f_4 is *not* twice continuously differentiable. (etc.)

8 Since every *positive* rational number is a period of f, we have

$$f(0) = f(q) \quad \text{for all} \quad q \in (0, \infty) \cap \mathbb{Q},$$

and also

$$f(-q) = f(-q + q) = f(0)$$

for all such q. Thus

$$f(x) = f(0) \quad \text{for all} \quad x \in \mathbb{Q}.$$

Setting $g(x) = f(0)$ (constant), we can apply the result of question 2 to deduce that $f(x) = f(0)$ for all $x \in \mathbb{R}$, so f is constant.

9 (a) We start by proving that $f(nx) = nf(x)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. To see this note, for $x \in \mathbb{R}$,

$$\begin{aligned} f(2x) &= f(x + x) = f(x) + f(x) = 2f(x) \\ f(3x) &= f(2x + x) = f(2x) + f(x) = 2f(x) + f(x) = 3f(x) \end{aligned}$$

etc: an easy induction argument gives $f(nx) = nf(x)$ for all $n \in \mathbb{N}$.
Now $f(1) = 1$, so the above gives us $f(n) = n$ for all $n \in \mathbb{N}$.
Now let $p, q \in \mathbb{N}$. Then

$$p = f(p) = f\left(q\left(\frac{p}{q}\right)\right) = qf\left(\frac{p}{q}\right) \quad \text{by above}$$

so that $f\left(\frac{p}{q}\right) = \frac{p}{q}$. This gives us $f(x) = x$ for all x in $\mathbb{Q} \cap (0, \infty)$.

(b) We can use continuity:

$$\begin{aligned} f(0) &= \lim_{n\to\infty} f\left(\tfrac{1}{n}\right) \\ &= \lim_{n\to\infty} \left(\tfrac{1}{n}\right) = 0. \end{aligned}$$

Or directly note

$$\begin{aligned} f(0) &= f(0+0) \\ &= f(0)+f(0) \\ &= 2f(0) \end{aligned}$$

so $f(0)=0$ (subtracting $f(0)$ from both sides).
Now

$$\begin{aligned} 0 = f(0) &= f(x+(-x)) \\ &= f(x)+f(-x) \quad (x \in \mathbb{R}) \end{aligned}$$

so that $f(-x) = -f(x)$ for all real x.

(c) From (a) and (b) we see that $f(x) = x$ for all $x \in \mathbb{Q}$. Thus, taking $g(x) = x$ we can apply the result of question 3 to see that $f(x) = x$ for all real x.

TRICKY EXERCISE! Show that there *are* some *discontinuous* functions $h : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying $h(1) = 1$ and $h(x+y) = h(x) + h(y)$ for all real x, y.

[If you learn about vector spaces over $\mathbb{Q}$, you can do this by taking a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$!]

10 (a)

$$\begin{aligned} \tfrac{d}{dx}(x^a) &= \tfrac{d}{dx}(\exp(a\log(x))) \\ &= \text{(chain rule)}\ \exp(a\log(x))\tfrac{d}{dx}(a\log(x)) \\ &= \text{(product rule)}\ \exp(a\log(x))\tfrac{a}{x} \\ &= \text{(standard)}\ a\exp(a\log(x))\exp((-1)\log(x)) \\ &= \text{(standard)}\ a\exp((a-1)\log(x)) \\ &= \text{(definition)}\ ax^{a-1}. \end{aligned}$$

(b) (i)

$$\begin{aligned} x^{\sqrt{x}} &= \exp(\sqrt{x}\log(x)) \\ \tfrac{d}{dx}(\exp(\sqrt{x}\log(x))) &= \exp(\sqrt{x}\log(x))\tfrac{d}{dx}(\sqrt{x}\log(x)) \text{ (chain rule)} \\ &= \exp(\sqrt{x}\log(x))\left(\tfrac{1}{2\sqrt{x}}\log(x) + \tfrac{\sqrt{x}}{x}\right) \\ &= x^{\sqrt{x}}\left(\tfrac{1}{2\sqrt{x}}\log(x) + \tfrac{1}{\sqrt{x}}\right). \end{aligned}$$

[Here we used the fact that $\sqrt{x} = x^{1/2}$, so $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.]
[An alternative method involves "logarithmic differentiation".]

(ii)

$$\begin{aligned} \tfrac{d}{dx}\left((\log(x))^{\cos(2x)}\right) &= \tfrac{d}{dx}\left(\exp(\cos(2x)\log(\log(x)))\right) \\ &= \exp(\cos(2x)\log(\log(x))))\left[-2\sin(2x)\log(\log(x)) + \cos(2x)\tfrac{1}{\log(x)}\tfrac{1}{x}\right] \\ &= (\log(x))^{\cos(2x)}\left[-2\sin(2x)\log(\log(x)) + \tfrac{\cos(2x)}{x\log(x)}\right]. \end{aligned}$$