G12RAN Real Analysis

Exercises 4: Solutions to questions 6-10

6 Suppose, for contradiction, that no such s exists. Then $f'(s) \neq 0$ for all $s \in (a,b)$. (*)

Note that f is differentiable on (a,b), and hence f is also continuous on (a,b).

STAGE I. We show that f must be $1-1$ on (a,b). Let $x, y \in (a,b)$ with $x < y$. Then we can apply the mean value theorem to f on $[x,y]$, and there must be a c in (x,y) with $f'(c) = \dfrac{f(y)-f(x)}{y-x}$. By (*) above, $f'(c) \neq 0$, and so $f(x) \neq f(y)$. This shows that f is injective on (a,b).

STAGE II. We saw on an earlier question sheet that every continuous, injective real-valued function on (a,b) must be strictly monotone. We are given in the question that $f'(c) < 0$ and $f'(d) > 0$. This is impossible if f is strictly monotone on (a,b) [do you know how to prove this?] and this gives us the desired contradiction.

Thus there must, after all, be an s in (a,b) with $f'(s) = 0$.

7 In this question you can use the mean value theorem either directly or indirectly!

Method 1. Set $f(x) = \log(1+x)$. Then f is continuous on $(-1,\infty)$, and is differentiable there, with $f'(x) = \dfrac{1}{1+x}$ (for x in $(-1,\infty)$). Also, $f(0) = \log(1) = 0$.

Let $x > 0$. Then, by the mean value theorem, there exists a c in $(0,x)$ with

$$\frac{f(x)-f(0)}{x-0} = f'(c)$$

i.e. (from above)

$$\frac{f(x)}{x} = \frac{1}{1+c}.$$

Since $c \in (0,x)$, we have $\dfrac{1}{1+x} < \dfrac{1}{1+c} < 1$ and so $\dfrac{1}{1+x} < \dfrac{f(x)}{x} < 1$,

$$\text{i.e.} \quad \frac{x}{1+x} < f(x) < x \quad (\text{since } x > 0)$$

as required.

Method 2. Each of the functions $\dfrac{x}{1+x}$, $\log(1+x)$ and x are 0 when $x = 0$.

If you differentiate each of these 3 functions and compare the derivatives, you can use the mean value theorem to say that, since

$$\frac{d}{dx}\left(\log(1+x) - \frac{x}{1+x}\right) > 0 \quad \text{for} \quad x > 0$$

and

$$\frac{d}{dx}(x - \log(1+x)) > 0 \quad \text{for} \quad x > 0,$$

it follows that these two functions are strictly increasing on $[0, \infty)$. The result then follows. [Exercise: check the details of this!]

8 (a) First rewrite $\frac{1}{\sin x} - \frac{1}{x}$ as $\frac{x - \sin x}{x \sin x}$, and consider the limit as $x \to 0+$. This is indeterminate of type "0/0" as $x \to 0+$.

Differentiating numerator and denominator gives

$$\frac{1 - \cos x}{x \cos x + \sin x},$$

which is still indeterminate of type "0/0" as $x \to 0+$.

Differentiating top and bottom again gives

$$\frac{\sin x}{2 \cos x - x \sin x}.$$

This tends to 0 as $x \to 0+$ by the algebra of limits, so l'Hôpital's rule applied twice gives

$$\lim_{x \to 0+} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = 0.$$

(b) This is not an indeterminate form! Since $1 + \sin(0) \neq 0$, the algebra of limits and continuity of the relevant functions gives

$$\lim_{x \to 0} \left(\frac{\cos x}{1 + \sin x} \right) = \frac{\cos(0)}{1 + \sin(0)} = 1.$$

Note that l'Hôpital's rule does not apply.

(c) Substituting $y = \frac{1}{x}$, we must investigate $\lim_{y \to 0+} (\cos(y))^{1/y^2}$.

If this limit exists, then so does the original limit, and with the same value. [This is one of the available definitions of $\lim_{x \to +\infty}$.]

Now the function exp is continuous, so if $\lim_{y \to 0+} (\log((\cos(y))^{1/y^2}))$ exists, we can take exp and see that the original limit exists (and is exp of the new limit). We look at the limit os $y \to 0+$ of

$$\log((\cos(y))^{1/y^2}) = \frac{\log(\cos(y))}{y^2}.$$

This is indeterminate of type "0/0" as $y \to 0+$ (note that $\cos(y) \to 1$ as $y \to 0+$).

Differentiating top and bottom gives $\dfrac{\frac{1}{\cos(y)}(-\sin(y))}{2y}$. We have $\cos(y) \to 1$ as $y \to 0+$ and $\dfrac{\sin(y)}{y} \to 1$ as $y \to 0+$ by l'Hôpital's rule, or noting that

$$\lim_{y \to 0} \left(\frac{\sin(y) - \sin(0)}{y} \right) = \sin'(0) = \cos(0) = 1.$$

So $\lim_{y\to 0+}\left(\dfrac{\frac{1}{\cos(y)}(-\sin(y))}{2y}\right) = -\dfrac{1}{2}$.

Applying l'Hôpital's rule and taking exp, the original limit exists and equals $\mathrm{e}^{-\frac{1}{2}} = \dfrac{1}{\sqrt{\mathrm{e}}}$.

9 We are required to prove that $\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0}$ exists and equals L. Set $F(x) = f(x) - f(0)$ and $G(x) = x$. Then, since f is continuous, $\lim_{x\to 0} F(x) = 0$, and of course $\lim_{x\to 0} G(x) = 0$ too. However, $F'(x) = f'(x)$ (n.b. $f(0)$ is a *constant*, and the derivative of a constant is 0) and $G'(x) = 1$, so $\lim_{x\to 0}\dfrac{F'(x)}{G'(x)}$ exists and equals L (given in question).

So l'Hôpital's theorem applies to give $\lim_{x\to 0}\dfrac{F(x)}{G(x)} = L$ too, i.e. $\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = L$, as required.

10 By the definition of $g'(0)$, we have

$$
\begin{aligned}
0 = g'(0) &= \lim_{x\to 0}\frac{g(x)-g(0)}{x-0} && \\
&= \lim_{x\to 0}\frac{g(x)}{x} && \text{(N.B. } g(0) = g'(0) = 0\text{)} \\
&= \lim_{x\to 0} f(x) && \text{(since } f(x) = \tfrac{g(x)}{x} \text{ for } x \neq 0\text{),} \\
&= f(0) && \text{(since } f \text{ is continuous).}
\end{aligned}
$$

So $f(0) = 0$.

To find $f'(0)$, note that

$$f'(0) = \lim_{x\to 0}\frac{f(x)-f(0)}{x} = \lim_{x\to 0}\frac{g(x)}{x^2}$$

(since $f(0) = 0$ and $f(x) = \dfrac{g(x)}{x}$ for $x \neq 0$).

To determine this latter limit, we can use l'Hôpital's rule. Certainly $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} x^2 = 0$. So we look at $\dfrac{g'(x)}{2x}$ (differentiating top and bottom). But $g'(0) = 0$, so $\lim_{x\to 0}\dfrac{g'(x)}{x} = g''(0) = 6$, and so $\lim_{x\to 0}\dfrac{g'(x)}{2x} = 3$. Thus, by l'Hôpital's rule, $\lim_{x\to 0}\dfrac{g(x)}{x^2} = 3$ too, and this gives $f'(0) = 3$.