G12RAN Real Analysis

Exercises 4: Solutions to questions 1-5

1 $f'(x) = 3x^2 + 1 + \sin x$. Now $3x^2 \geq 0$ and $1 + \sin x \geq 0$, so it is clear that $f'(x) \geq 0$ for all x. Since the only solution to $3x^2 = 0$ is $x = 0$, and then $1 + \sin x = 1 > 0$, it follows that $f'(x) > 0$ for all x, as required. This tells you that the function f must be strictly increasing. (See the printed notes for more details).

2 (i) Since f is continuous at 0 and $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$, we must have

$$f(0) = \lim_{n\to\infty} f\left(\frac{1}{n}\right) = 0,$$

since we are given that $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$.

(ii) By part (i) we have $f(0) = 0$ and so

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \left(\frac{f(x) - f(0)}{x - 0}\right) \\ &= \lim_{x\to 0} \left(\frac{f(x)}{x}\right). \end{aligned}$$

Thus, since $\frac{1}{n} \neq 0$ and $\frac{1}{n} \to 0$ as $n \to \infty$,

$$f'(0) = \lim_{n\to\infty} \left(\frac{f\left(\frac{1}{n}\right)}{\frac{1}{n}}\right) = 0,$$

because $f\left(\frac{1}{n}\right) = 0$ for all $n \in \mathbb{N}$.

3 $f(x) = x \arcsin(x) + \sqrt{1 - x^2}$. The function f is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$, with derivative

$$\begin{aligned} f'(x) &= \arcsin(x) + \frac{x}{\sqrt{1 - x^2}} + \frac{1}{2}(-2x)/\sqrt{1 - x^2} \\ &= \arcsin(x). \end{aligned}$$

We check *endpoints* and *stationary points in the range.*

For $-1 < x < 1$,

$$\begin{aligned} f'(x) = 0 &\Leftrightarrow \arcsin(x) = 0 \\ &\Leftrightarrow x = 0 \end{aligned}$$

When $x = 0$, $f(x) = 1$; when $x = -1$, $f(x) = \frac{\pi}{2}$ (since $\arcsin(-1) = -\frac{\pi}{2}$); when $x = +1$, $f(x) = \frac{\pi}{2}$ (since $\arcsin(1) = \frac{\pi}{2}$).

Since $\frac{\pi}{2} > 1$, the greatest value of $f(x)$ in this range is $\frac{\pi}{2}$, and the least value is 1.

[In fact f is strictly increasing on $[0, 1]$ and $f(-x) = f(x)$; f is an "even" function of x. Functions which instead satisfy $f(-x) = -f(x)$ are called "odd" functions of x.]

4 The answer is *no*. To prove this we use the mean value theorem. Let f be a differentiable function from $\mathbb{R}$ to $\mathbb{R}$ such that $f'(x) > 1$ for all $x > 0$. Then, by the mean value theorem, for each $x > 0$ there is a $c_x \in (0, x)$ such that

$$f'(c_x) = \frac{f(x) - f(0)}{x - 0}.$$

Now $f'(c_x) > 1$, by assumption, and so $\dfrac{f(x) - f(0)}{x - 0} > 1$ for all $x > 0$. Thus it is impossible for us to have $\lim_{x \to 0+} \dfrac{f(x) - f(0)}{x - 0} = 0$ and so it is also impossible to have $f'(0) = 0$. [In fact, since we know $f'(0)$ exists, the above shows that $f'(0) \geq 1$.]

5 We have $f(x) = \cos(\log(x))$, $f'(x) = -\sin(\log(x))/x$ and so

$$\left|f'(x)\right| \leq 1 \text{ for } x \in (1, \infty). \qquad (*)$$

Set $A = 1$. By $(*)$ above and a standard result in the notes (using the MVT) we have, for all $x, y \in (1, \infty)$,

$$|f(x) - f(y)| \leq A\,|x - y|\,.$$

So f satisfies the condition for Lipschitz continuity on this interval with constant $A = 1$, as required.