G12RAN Real Analysis: Exercises 5

You should attempt questions 1 to 3 in the fifth problem class. If you finish early then try question 4. **You should make sure that you are familiar with all the concepts mentioned in questions 1-4 before the problem class.**

Answers to questions 7 and 12 should be handed in to Dr Feinstein's pigeonhole at the top of the main stairs by the end of term (Wednesday 11/12/01)

1. Apply Taylor's theorem to the function $f(x) = \sin(x)$ to show that $|\sin(x) - x| \le |x|^3/6$ for all $x \in \mathbb{R}$. [You should find the remainder term after using the terms in the Taylor series up to and including the term in x^2.]

2. Define $f : (0, \infty) \to \mathbb{R}$ by $f(x) = \sqrt{x}$.

(i) You should know that this function is differentiable. What is its derivative?

(ii) Show that the function f is *not* Lipschitz continuous on $(0, \infty)$. [Hint: by a standard result in the notes, this is the same as showing that $f'(x)$ is unbounded on this interval.]

(iii) Show that, for all a, b in $[0, \infty)$, we have

$$\sqrt{a+b} \le \sqrt{a} + \sqrt{b}.$$

(iv) Prove that f is uniformly continuous on $(0, \infty)$. [Hint: use (iii) to show that when $x \ge y > 0$ we have $0 \le f(x) - f(y) \le \sqrt{x-y}$. Deduce that for *all* x, y in $(0, \infty)$ we have

$$|f(x) - f(y)| \le \sqrt{|x-y|}. \qquad (*)$$

Now take sequences (x_n), (y_n) in $(0, \infty)$ with $\lim_{n\to\infty} |x_n - y_n| = 0$. What does $(*)$ tell you about $|f(x_n) - f(y_n)|$?]

(Because of $(*)$, the function $f(x) = \sqrt{x}$ is said to satisfy a *Lipschitz condition of order* $1/2$.)

3. Find the Taylor series $T(x, \pi/4)$ for $\cos(x)$ (this means the Taylor series for $\cos(x)$ in powers of $(x-\pi/4)$ and has the form

$$a_0 + a_1(x - \pi/4) + a_2(x - \pi/4)^2 + \ldots$$

so you just need to find the coefficients a_n. Use the formula in the notes. Remember: you must work in radians).

4. Define f from $\mathbb{R}$ to $\mathbb{R}$ by $f(x) = \exp(-1/x^2)$ when $x \ne 0$, while $f(0) = 0$. Prove by induction on n that, for each $n \in \mathbb{N}$, f is n-times differentiable on $\mathbb{R}$, and that there is a polynomial p_n such that $f^{(n)}(x) = p_n(1/x)\exp(-1/x^2)$ for $x \ne 0$, while $f^{(n)}(0) = 0$.

5. Let f be a real-valued function on (a, b) and suppose that F_1, F_2 are both antiderivatives (primitives) for f on (a, b) (i.e. $F_1' = F_2' = f$ on (a, b)). Prove that $F_1 - F_2$ is constant on (a, b). (Hint: apply the MVT or some similar standard result to the function $F_1 - F_2$.)

6. Just using the definitions of upper sum, lower sum etc., and NOT using calculus, prove that if c is a constant then, for $a < b \in \mathbb{R}$, $\int_a^b c \, \mathrm{d}x = c(b - a)$ [Strictly speaking, define f on $[a, b]$ by $f(x) = c$: prove that $\int_a^b f(x) \, \mathrm{d}x = c(b - a)$.]

7. For $x > -1$, let $f(x) = \log(1+x)$.
(a) For each $n \in \mathbb{N}$, find a formula for the nth derivative of f, $f^{(n)}(x)$. [You need not give a full proof by induction, but you should show in your working that if you differentiate what you claim is $f^{(n)}(x)$ that you do get what you claim is $f^{(n+1)}(x)$.]
(b) Find the Maclaurin series for f.

8. Let f be the restriction to $[0,1]$ of the characteristic function of $\mathbb{Q}$, so that, for $x \in [0,1]$, $f(x) = 1$ if $x \in \mathbb{Q}$, and $f(x) = 0$ otherwise.
(i) Prove that every Riemann upper sum for f is 1 and that every Riemann lower sum for f is 0.
(ii) Deduce that f is not Riemann integrable on $[0,1]$.
[To integrate this function a more powerful integration method is needed. In G1CMIN you will see that, using the *Lebesgue integral*, the integral of f is zero because f is only non-zero at countably many points]

9. Define $f : [-1,1] \longrightarrow \mathbb{R}$ by $f(0) = 1$, while $f(x) = 0$ for $x \neq 1$. (i) Show that f is Riemann integrable on $[-1,1]$, and that $\int_{-1}^{1} f(x)\ \mathrm{d}x = 0$. (ii) Show that there is no antiderivative (primitive) for f on $(-1,1)$, i.e. there is no differentiable function F on $(-1,1)$ such that $F' = f$ on $(-1,1)$.

10. Let $a < b \in \mathbb{R}$ and let f be a non-negative, continuous function defined on $[a,b]$. Suppose that $\int_a^b f(x)\ \mathrm{d}x = 0$. Prove that f must be constantly 0 on $[a,b]$. [Note: this is only true because f is continuous, as question 8 shows. Hint: look at $F(x) = \int_a^x f(t)\ \mathrm{d}t$].

11. Let f, g be Riemann integrable functions on a closed interval $[a,b]$. Suppose that $f(x) \le g(x)$ for all $x \in [a,b]$. Prove that $\int_a^b f(x)\ \mathrm{d}x \le \int_a^b g(x)\ \mathrm{d}x$. Deduce that $|\int_a^b f(x)\ \mathrm{d}x| \le \int_a^b |f(x)|\ \mathrm{d}x$. (You may assume that the functions $|f(x)|$ and $-|f(x)|$ are Riemann integrable on $[a,b]$).

12. Show that the following limits exist, and evaluate them. (You may assume the Fundamental Theorem of Calculus, which allows you to integrate *continuous* functions on *closed* intervals in the usual way, but does not directly tell you about other types of integrals.)
(i) $\lim_{\epsilon \to 0+} \int_\epsilon^1 1/\sqrt{x}\ \mathrm{d}x$; (ii) $\lim_{x \to \infty} \int_1^x \exp(-t)\ \mathrm{d}t$.
[When such limits exist they are called *convergent improper Riemann integrals*. The first is often denoted by $\int_0^1 1/\sqrt{x}\ \mathrm{d}x$, even though $1/\sqrt{(x)}$ is not defined when $x = 0$. The second is often denoted by $\int_1^\infty \exp(-t)\ \mathrm{d}t$].