G12RAN Real Analysis

Exercises 5: Solutions to questions 5-12

5 This is almost immediate from the MVT. Since $F_1' = F_2'$, we have $\frac{d}{dx}(F_1 - F_2) = 0$ on (a, b), and so the mean value theorem tells us that $F_1 - F_2$ is constant on (a, b).

6 Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, where $a = x_0 < x_1 < \cdots < x_n = b$.

Then $m_k(f) = M_k(f) = c$ for $1 \le k \le n$, so

$$L(P, f) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) = \sum_{k=1}^{n} c(x_k - x_{k-1}) = c(b - a)$$

and similarly

$$U(P, f) = c(b - a).$$

Then the lower integral $\underline{\int_a^b} f(x)\, dx$ is the supremum (over all partitions) of $L(P, f)$, so this is also $c(b - a)$, and similarly for the upper integral: $\overline{\int_a^b} f(x)\, dx$ is the infimum (over all possible partitions P) of $U(P, f)$, and this is also $c(b - a)$. Thus $\underline{\int_a^b} f(x)\, dx = \overline{\int_a^b} f(x)\, dx$, so f is Riemann integrable on $[a, b]$ and $\int_a^b f(x)\, dx = c(b - a)$.

[ALTERNATIVELY: When $P = \{a, b\}$, $U(P, f) = L(P, f) = c(b - a)$. But any partition of $[a, b]$ must be a refinement of this one, and the usual inequalities force the lower sum and upper sum to equal $c(b - a)$ again.]

7 (a)

$$\begin{aligned} f^{(0)}(x) &= f(x) = \log(1 + x), \\ f^{(1)}(x) &= f'(x) = \tfrac{1}{1+x}, \\ f^{(2)}(x) &= \tfrac{-1}{(1+x)^2}, \\ f^{(3)}(x) &= \tfrac{2}{(1+x)^3} \end{aligned}$$

and now an easy induction shows that

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n} \quad (n = 1, 2, 3, \ldots).$$

As stated in the question, you should show in your working that the derivative of $\frac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ is $\frac{(-1)^n n!}{(1+x)^{n+1}}$: this follows immediately from the fact that the derivative of the function $(1+x)^{-n}$ is $(-n)(1 + x)^{-n-1}$.

(b) The Maclaurin series for f is $\sum_{k=0}^{\infty} a_k x^k$, where $a_k = \frac{f^{(k)}(0)}{k!}$.
So, here,

$$a_k = (-1)^{k-1}\frac{(k-1)!}{k!} = \frac{(-1)^{k-1}}{k}$$

for $k = 1, 2, \ldots$.
$a_0 = f(0) = 0$.
Thus the Maclaurin series for $\log(1+x)$ is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \ldots.$$

[In fact this series does converge to $\log(1+x)$ for $-1 < x < 1$. See books if interested!]

8 This comes from the fact that the rationals and the irrationals are dense.

(a) Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[0,1]$ with $0 = x_0 < x_1 < \cdots < x_n = 1$. Then because every interval $[x_{k-1}, x_k]$ contains infinitely many rationals and infinitely many irrationals, we obtain

$$m_k(f) = 0 \quad \text{and} \quad M_k(f) = 1 \quad \text{for} \quad 1 \le k \le n.$$

Thus

$$L(P,f) = \sum_{k=1}^{n} m_k(f)(x_k - x_{k-1}) = 0$$

while

$$U(P,f) = \sum_{k=1}^{n} M_k(f)(x_k - x_{k-1}) = \sum_{k=1}^{n}(x_k - x_{k-1}) = 1,$$

as required.

(b) It follows that $\underline{\int_0^1} f(x)\, dx = 0$ and $\overline{\int_0^1} f(x)\, dx = 1$.
Since the lower and upper integrals are different, f is NOT Riemann integrable on $[0,1]$.

9 (a) It is clear that every lower sum for f is zero, so the lower integral $\underline{\int_{-1}^1} f(x)\, dx = 0$ also.
The upper sums for f are each > 0, but we show that the inf of the upper sums is 0.
Let $\varepsilon > 0$. Choose $m \in \mathbb{N}$ with $m > \frac{2}{\varepsilon}$. Consider the partition $\{x_0, x_1, \ldots, x_n\}$ of $[-1,1]$ where $n = 2m$ and $x_k = -1 + \frac{k}{m}$ $(0 \le k \le 2m)$. Then $M_k(f) = 0$ except for $M_m(f) = M_{m+1}(f) = 1$.
So

$$\begin{aligned} U(P,f) &= \textstyle\sum_{k=0}^{n} M_k(f)(x_k - x_{k-1}) \\ &= \tfrac{2}{m} < \varepsilon. \end{aligned}$$

Thus $\overline{\int_{-1}^1} f(x)\, dx < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we obtain $\overline{\int_{-1}^1} f(x)\, dx \le 0$, and we deduce that we must have $\underline{\int_{-1}^1} f(x)\, dx = \overline{\int_{-1}^1} f(x)\, dx = 0$. The result follows.
[An alternative approach is to use Riemann's criterion.]

(b) Suppose that F is an antiderivative for f on $(-1,1)$. (We shall obtain a contradiction.) Set $G(x) = 2F(x) - x$. Then for $x \in (-1,1)$

$$\begin{aligned} G'(x) &= 2F'(x) - 1 \\ &= 2f(x) - 1 \\ &= \begin{cases} -1 & x \neq 0, \\ 1 & x = 0. \end{cases} \end{aligned}$$

But, by sheet 4, question 6, since $G'(x)$ takes positive and negative values on $(-1,1)$, there should be a c in $(-1,1)$ with $G'(c) = 0$. There is no such c, so we have a contradiction, showing that no such antiderivative F can exist.

10 We have

$$\begin{aligned} 0 \leq F(x) &= \int_a^x f(t)\ dt \\ &\leq \int_a^x f(t)\ dt + \int_x^b f(t)\ dt \\ &= \int_a^b f(t)\ dt = 0, \end{aligned}$$

for $a \leq x \leq b$, so $F(x) = 0$ for $a \leq x \leq b$. But, by the fundamental theorem of calculus, for $x \in (a,b)$, $f(x) = F'(x) = 0$. Finally, the continuity of f forces $f(a) = \lim_{x \to a+} f(x) = 0$, and similarly $f(b) = 0$.

11 Since every lower sum $L(P,f)$ is clearly less than or equal to the corresponding lower sum for g, $L(P,g)$, it follows immediately that

$$\int_a^b f(x)\ dx = \sup_P L(P,f) \leq \sup_P L(P,g) = \int_a^b g(x)\ dx$$

(where P runs through all partitions of $[a,b]$).

Now we have $-|f(x)| \leq f(x) \leq |f(x)|$ for all x in $[a,b]$, so

$$-\int_a^b |f(x)|\ dx \leq \int_a^b f(x)\ dx \leq \int_a^b |f(x)|\ dx$$

and so

$$\left| \int_a^b f(x)\ dx \right| \leq \int_a^b |f(x)|\ dx,$$

as claimed.

[Here we used the fact that

$$\int_a^b -|f(x)|\ dx = -\int_a^b |f(x)|\ dx.$$

In fact it is true that for any real number α and Riemann integrable function h on $[a,b]$ that

$$\int_a^b \alpha h(x)\ dx = \alpha \int_a^b h(x)\ dx.]$$

[An alternative method for the first part is to look at the function $f(x) - g(x)$.]

12 From the fundamental theorem of calculus it follows that integration agrees with antidifferentiation, at least when integrating continuous functions on closed intervals.

(a)

$$\begin{aligned}\int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx &= \textstyle\int_\varepsilon^1 x^{-1/2}\,dx \\ &= \left[\frac{x^{1/2}}{1/2}\right]_\varepsilon^1 \quad \text{(usual notation)} \\ &= 2(1-\varepsilon^{1/2})\end{aligned}$$

and this tends to 2 as $\varepsilon \to 0+$, so

$$\lim_{\varepsilon\to 0+} \int_\varepsilon^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

(b)

$$\begin{aligned}\textstyle\int_1^x \mathrm{e}^{-\mathrm{t}}\,\mathrm{dt} &= [-\mathrm{e}^{-\mathrm{t}}]_1^{\mathrm{x}} \\ &= \mathrm{e}^{-1} - \mathrm{e}^{-\mathrm{x}} \\ &\to \mathrm{e}^{-1} \quad \text{as} \quad \mathrm{x} \to \infty,\end{aligned}$$

so

$$\lim_{x\to\infty} \int_1^x \mathrm{e}^{-\mathrm{t}}\,\mathrm{dt} = \frac{1}{\mathrm{e}}.$$

So, as "improper Riemann integrals",

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2 \quad \text{and} \quad \int_1^\infty \exp(-t)\,dt = \frac{1}{\mathrm{e}}.$$

[See books for more information on improper integrals.]