G12RAN BLOW BY BLOW 2002-2003

The following material has been covered in lectures so far.

Lecture 1: *Chapter 1. Properties of the real numbers* **I. Review of notation, definitions and results from earlier modules** The sets of natural numbers, integers, rational numbers. The irrational numbers. Other subsets of $\mathbb{R}$ including intervals. Convergence of sequences.

Lecture 2: Divergence of sequences. Further revision of results about sequences including the Algebra of Limits and the Sandwich Theorem. The completeness of $\mathbb{R}$ (existence of supremum/infimum for non-empty, bounded sets).

Lecture 3: Monotone sequences and the Monotone Sequence Theorem (MST). Density of the rationals and the irrationals in $\mathbb{R}$: (as in G1ALIM) every open interval (a, b) in $\mathbb{R}$ contains infinitely many rational numbers and infinitely many irrational numbers. **II. Further properties** The nested intervals theorem: Given non-empty closed intervals $[a_n, b_n]$ such that, for all $n \in \mathbb{N}$, $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$, then there must be at least one point c common to all of these closed intervals. If the lengths of the intervals involved tend to 0, then there is *exactly* one such point c. Failure of the nested intervals theorem for open intervals (exercise in first problem class). Intersections and unions of infinitely many sets: notation.

Lecture 4: Examples of infinite intersections and unions.
Chapter 2. Functions and sets: Functions, graphs and Cartesian products. Examples of functions between subsets of $\mathbb{R}$ including characteristic functions. Injections (injective/1-1 functions), surjections (surjective/onto functions) and bijections (bijective functions) revised. Addition and multiplication for real-valued functions (pointwise operations). Composition of functions ($f \circ g$ or $f(g)$). Inverse functions for bijections. Two sets have the same cardinality if there is a bijection between them. An example of a bijection between $\mathbb{N}$ and $\mathbb{Z}$: $\mathbb{Z}$ is countable. Uncountable sets include $\mathbb{R}$, $[0, 1]$, $[0, 1)$ (proofs next time). Working definition for first problem class: a set is countable if it is empty or else there is a sequence of elements which includes (uses up) all of the elements of the set. Equivalence of this definition to the standard definition.

Lecture 5: Which real numbers have two different decimal expansions? Uncountability of $[0, 1)$ (Cantor diagonalization argument). Uncountability of $\mathbb{R}$ follows. Unions of two countable sets and countability of $\mathbb{Q}$ were covered in the first problem class (solutions available from module web page). Uncountability of $\mathbb{R} \setminus \mathbb{Q}$.

Lecture 6: Other standard results about countability: images under functions of countable sets are countable (i.e. if X is a countable set, Y is a set and f is a surjection from X to Y then Y must also be countable, proof left as an exercise); if there is an injection f from a set Y into a countable set X then Y must also be countable; in particular, subsets of countable sets are countable (details an exercise). $\mathbb{N} \times \mathbb{N}$ is countable. The product of two countable sets and countable unions of countable sets are done as the first homework assignment.
Chapter 3. Limit values for functions: Punctured neighbourhoods. Introductory examples illustrating the two types of one-sided limits for functions (formal definitions next lecture): $\lim_{x\to a-} f(x)$ and $\lim_{x\to a+} f(x)$. These one-sided limits may or may not exist. When both exist, they may be different. The value of the function at the point a is irrelevant here, and need not even be defined.

Lecture 7: Definition in terms of sequences of two-sided limits for functions defined on punctured neighbourhoods of a point. Careful consideration of the function $\sin(1/x)$ to show two methods of establishing when limits do not exist: finding one suitable convergent sequence (x_n) where the sequence of function values $(f(x_n))$ does not converge at all, or finding two different appropriate sequences (x_n) giving different values for $\lim_{n\to\infty}(f(x_n))$. Definition in terms of sequences of the one-sided limit $\lim_{x\to a+}$.

Lecture 8: Definition in terms of sequences of the one-sided limit $\lim_{x\to b-}$. Example of standard proof structure: direct proof that $\lim_{x\to 3-}(x^2) = 9$ using the algebra of limits for sequences. Connection between one-sided and two-sided limits. Defininitions and examples of some further concepts: convergence of $f(x)$ as $x \to \pm\infty$. Divergence of functions to $\pm\infty$ as x approaches a in the various possible ways.

Lecture 9: Feedback on student homework. Brief discussion of the four kinds of monotone functions.
Chapter 4. Sequences and continuous functions: Motivation: why do we need to prove intuitively obvious facts? Continuity and discontinuity of real-valued functions defined on intervals in terms of one/two sided limits.

Lecture 10: Continuity in terms of sequences. Discussion of the different ways in which a function can fail to be continuous at a point. Standard functions are continuous where they are defined. Sequences of points in the interval I which converge to some point outside I are not important when checking continuity. Subsequences of sequences. Statement ond discussion of the Bolzano-Weierstrass theorem and some related examples.

Lecture 11: Proof of the Bolzano-Weierstrass theorem. Discussion and proof of the boundedness theorem.

Lecture 12: New continuous functions from old: sums, products and quotients of continuous functions. Composition of continuous functions. Discussion and proof of the the intermediate value theorem.

Lecture 13: Feedback on student homework. Further discussion of the implications of the boundedness theorem and the intermediate value theorem: The continuous image of an interval is an interval. The continuous image of a closed and bounded interval is a closed and bounded interval.
Chapter 5. Differentiability: Differentiability at a point, interpretation in terms of limiting gradients. Problems with the function $f(x) = |x|$. Functions which are differentiable on intervals, including closed intervals $[a, b]$ (one-sided derivatives defined at the endpoints) and $\mathbb{R}$ (n.b. $\mathbb{R}$ is an interval).

Lecture 14: Differentiable functions must be continuous, but continuous functions need not be differentiable. In fact there are functions which are continuous everywhere but differentiable nowhere. All the usual standard functions (constant functions, polynomials, rational functions, trigonometric functions etc.) are differentiable where they are defined. (See books for details: standard facts about these functions may be assumed.) The algebra of derivatives: sums, products and quotients of differentiable functions are differentiable (avoiding division by 0). Proof of product rule. Proofs of sum rule and quotient rule left as an exercise. The chain rule for differentiation: standard false proof given. Exercise: find the mistake!

Lecture 15: Chord functions. Correct proof of the chain rule. Statement, discussion and proof of Rolle's Theorem. Reminder: how to find the greatest and least values of a function on a closed interval (example in problem class). Statement and brief discussion of the Mean Value Theorem.

Lecture 16: Proof of the MVT. Implications of the MVT, including connections with monotonicity. Definition of Lipschitz continuity. Boundedness of f' on an interval implies that f is Lipschitz continuous there. For differentiable functions, the converse is also true.

Lecture 17: Feedback on student homework.

Chapter 6. L'Hôpital's rule and Taylor's theorem: Higher order derivatives: n times differentiable functions, n times continuously differentiable functions. Infinitely differentiable functions. Limits of quotients of functions $f(x)/g(x)$. When possible, use the algebra of limits! Indeterminate forms of type '0/0' and of type '∞/∞'. Various forms of L'Hôpital's rule stated. Examples of applications of L'Hôpital's rule, and some alternative methods for easy cases.

Lecture 18: More general statement of L'Hôpital's rule (in terms of generalised notion of limit, 'lim', which includes possible divergence of the function to ∞). Proof of one easy version of L'Hôpital's rule. Students should know statements of all versions of L'Hôpital's rule and be able to use them, but the only version whose proof is examinable is the one proved in lectures. Taylor series and Maclaurin series. Examples of functions where the Taylor (or Maclaurin) series gives: (i) the original function, as hoped (e.g. $\exp(x)$) (ii) the constant function 0 (an infinitely differentiable which is not constant but all of whose derivatives at 0 are 0).

Lecture 19: Taylor's theorem and its application to error estimates (e.g. estimation of $\cos(0.1)$). The proof of Taylor's theorem is not examinable, but the statement is, as are its applications. The generalized second derivative test.

Lecture 20: *$\epsilon-\delta$ definitions and Uniform Continuity:* Short additional section, including the definition of uniform continuity and the connections between continuity, uniform continuity and Lipschitz continuity. Proof of the fact that every continuous function on a closed and bounded interval is uniformly continuous.

Lecture 21: Student opinion forms issued.

Chapter 7. Integration: **The statements and applications of results in this section are all examinable but the only proofs which are examinable are the ones given in lectures.** Brief discussion of antiderivatives (also called primitives) and the informal notion of area under the curve. The main result of this section is the Fundamental Theorem of Calculus which implies that every continuous function has an antiderivative. Area under the curve does not appear to make much sense for the characteristic function of $\mathbb{Q}$. Brief discussion of the idea behind Riemann integration: using rectangles to estimate areas from above and below. Partitions of intervals. Riemann upper and lower sums for a bounded function f corresponding to a given partition P.

Lecture 22: Refinements of partitions. Each Riemann lower sum $L(P, f)$ is less than or equal to every Riemann upper sum $U(Q, f)$. The Riemann upper and lower integrals. Riemann integrable functions (where the lower and upper Riemann integrals are the same) and the Riemann integral of such functions (equal to both the Riemann upper and lower integrals in this case). Riemann's criterion for integrability. Continuous real-valued functions on closed and bounded intervals are Riemann integrable. Other elementary facts about Riemann integration. Statement and brief discussion (but not proof) of the (first) fundamental theorem of calculus: if f is continuous on $[a, b]$, define $F(x)$ by

$$F(x) = \int_a^x f(t)\ \mathrm{d}t.$$

Then $F'(x) = f(x)$ for all $x \in (a, b)$. (For the proof, which is fairly straightforward, see, for example, Professor Langley's notes.) Thus continuous functions on intervals always have antiderivatives. Improper Riemann integrals: see final question sheet.