Uniform continuity

In this section we compare the condition of continuity with the stronger condition of uniform continuity, which may be thought of as lying between the conditions of continuity and Lipschitz continuity. First recall the $\epsilon - \delta$ definition of continuity.

It is an exercise (or see books) to prove that this definition is equivalent to the definitions we have worked with in lectures. (Details available on request.)

Definition Let A, B be subsets of $\mathbb{R}$ and let f be a function from A to B.
Let $a \in A$. Then f is *continuous at* a if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all x in A with $|x - a| < \delta$ we have $|f(x) - f(a)| < \epsilon$ (so, for $x \in A \cap (a - \delta, a + \delta)$ we have $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$).

Now we look at the definition of uniform continuity.

Definition Let A, B be subsets of $\mathbb{R}$ and let f be a function from A to B. Then f is uniformly continuous if, for all $\epsilon > 0$, there is a $\delta > 0$ such that whenever x, x' are in A and $|x - x'| < \delta$ then $|f(x) - f(x')| < \epsilon$.

This definition looks very similar to the definition of continuity. The difference is that with continuity you don't expect the same δ to work for different points a (see the definition above). With uniform continuity, the same δ is valid at all points a of A simultaneously. This is why uniformly continuous functions must be continuous, but continuous functions need not be uniformly continuous (in general).

A typical example of a continuous function which is not uniformly continuous is to take $A = B = \mathbb{R}$ and set $f(x) = x^2$. No matter how small δ is, you can find real numbers x, x' within δ of each other and such that $|f(x) - f(x')| \geq 1$, so the uniform continuity condition fails when $\epsilon = 1$ (in fact it fails for ALL $\epsilon > 0$, but only one failure is needed).

Equivalently, in terms of sequences, a function f is uniformly continuous from A to B if and only if the following condition holds: whenever (x_n), (y_n) are sequences of elements of A such that $\lim_{n\to\infty}(x_n - y_n) = 0$ then we also have $\lim_{n\to\infty}(f(x_n) - f(y_n)) = 0$.

Again this looks similar to continuity, but note that the sequences (x_n), (y_n) need not themselves converge to elements of A. For example, for the function $f(x) = x^2$, you can take $x_n = n$ and $y_n = (n^2 + 1)/n$. Then $\lim_{n\to\infty}(x_n - y_n) = 0$, but $f(x_n) - f(y_n)$ does not tend to 0 as $n \to \infty$. So f is not uniformly continuous from $\mathbb{R}$ to $\mathbb{R}$.

In fact it is useful to note two different ways of defining what it means for f to *not* be uniformly continuous.

Proposition Let A, B be subsets of $\mathbb{R}$ and let f be a function from A to B. Then the following conditions on f are equivalent.
(i) The function f is not uniformly continuous from A to B.
(ii) There is at least one pair of sequences (x_n), (y_n) of elements of A such that $\lim_{n\to\infty}(x_n - y_n) = 0$ but such that the sequence $(f(x_n) - f(y_n))$ does not converge to 0.
(iii) There is a positive number ϵ and at least one pair of sequences (x_n), (y_n) of elements of A such that $\lim_{n\to\infty}(x_n - y_n) = 0$ but such that for all n in $\mathbb{N}$ we have $|f(x_n) - f(y_n)| \geq \epsilon$.

Note that, in (ii), it is possible that the $(f(x_n) - f(y_n))$ does not converge at all. It would be *incorrect* to write $\lim_{n\to\infty}(f(x_n) - f(y_n)) \neq 0$. It is, however, acceptable to write $f(x_n) - f(y_n) \not\to 0$.

Perhaps surprisingly, if you restrict attention to closed intervals, continuity and uniform continuity turn out to be equivalent.

Theorem (Uniform Continuity Theorem) Let a, b be real numbers with $a < b$, and let f be a continuous function from $[a, b]$ to $\mathbb{R}$. Then f is uniformly continuous on $[a, b]$.