Module code: G1CMIN **Module name: Measure and integration**
Session: 2000-2001 **Credits: 15**
Module Lecturer: Dr J.F. Feinstein **Other staff involved: None**

Key concepts and results

• The extended real line, $\overline{\mathbb{R}}$: arithmetic, sequences and series, inf, sup, lim inf and lim sup. (Lectures 2–4)
• The topology of $\mathbb{R}$ and $\overline{\mathbb{R}}$: open sets and closed sets, continuous functions. (Lectures 4–5 + printed notes)
• Collections of sets: semi-rings, rings, fields and σ-fields. Half-open intervals, elementary figures in $\mathbb{R}$ and the Borel subsets or $\mathbb{R}$ and of $\overline{\mathbb{R}}$. (Lectures 5–9, Lecture 12, question sheets and printed notes)
• Introduction to theory of measures: hoped-for properties of Lebesgue measure; a non-measurable subset of $[0, 1]$. (Lectures 9-10)
• Measures on collections of sets: measures on semi-rings, rings and fields. (Lectures 10-11)
• Properties of measures on rings: countable additivity, finite additivity, monotonicity, countable subadditivity, continuity properties. (Lectures 11-12 and printed notes)
• Measurable spaces and measure spaces: counting measure, Lebesgue measure. Properties that hold almost everywhere. (Lecture 12 and throughout the rest of the module)
• Definitions and standard properties of Lebesgue outer measure, λ^*, and Lebesgue measure, λ (for subsets of $\mathbb{R}$): extension of notion of length of intervals, translation invariance, regularity. (Lecture 13, Lecture 27, Lecture 32, question sheets and printed notes)
• Restrictions of σ-fields to subsets. (Lecture 13)
• Equivalence of functions (almost everywhere equality). (Lecture 13)
• Simple functions and their properties, measurable functions (several equivalent definitions). (Lectures 14-16)
• Pointwise convergence of functions and uniform convergence of functions. (Tutorial session, printed notes, question sheets and throughout module).
• Measurable functions are closed under inf, sup, lim inf and lim sup, pointwise limits, sums (when defined), products, and taking positive and negative parts. (Lectures 16-18 and Lecture 20)
• Monotone approximation from below of non-negative measurable functions by non-negative simple measurable functions. (Lecture 19)
• Definition of the integral: integral of non-negative, simple, measurable functions; integral of non-negative measurable functions; integral (when defined) of $\overline{\mathbb{R}}$-valued measurable functions. (Lectures 20-22, Lecture 24).
• The Monotone Convergence Theorem and its corollaries. (Lectures 22-23)
• Fatou's Lemma. (Lecture 23)
• L^1 spaces. (Lecture 24)
• The Dominated Convergence Theorem. (Lecture 25)
• Sets of measure zero and how to discount them when integrating. (Lectures 25-26)

• The connection between the Lebesgue integral and the Riemann integral. (Lecture 26)

• Extension of measures: extension of a measure from a semi-ring to a ring, the theory of outer measures and measurable sets, extension of measures from a semi-ring or ring to a sigma-field. (Lectures 27-32)

• The construction of Lebesgue measure on the real line. (Lectures 27-32)

Detailed blow-by-blow account

Lecture 1: *Chapter 0. Introduction* General description of content and motivation for the module (length and area, connections with integrals, modes of convergence of functions, convergence theorems for integrals, formal manipulation of ∞). Book recommendations, especially Rudin's *Real and complex analysis.*

Lecture 2: *Chapter 1. The extended real line* The extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$, a totally ordered set. Standard facts and definitions explained: most details left as exercises (some on question sheet 1). EVERY subset E of $\overline{\mathbb{R}}$ has an infimum and a supremum in $\overline{\mathbb{R}}$, denoted by $\inf(E)$ and $\sup(E)$ respectively. Sequences in $\overline{\mathbb{R}}$: the limit infimum and limit supremum of a sequence ($\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$). For a sequence $(x_n) \subseteq \mathbb{R}$ and $x \in \mathbb{R}$, $\lim_{n\to\infty}(x_n) = x$ if and only if $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$. This enables us to give a definition of convergence in $\overline{\mathbb{R}}$ extending the previous notion of convergence of sequences in $\mathbb{R}$.

Lecture 3: Convergent sequences in $\overline{\mathbb{R}}$. Sandwich theorem. Equivalent definitions of convergence to $\pm\infty$ in $\overline{\mathbb{R}}$. Arithmetic in $\overline{\mathbb{R}}$. The minus operator $x \mapsto -x$. Addition and subtraction (where possible) and multiplication in $\overline{\mathbb{R}}$. Problems with the cancellation law for addition (only real numbers may be cancelled).

Lecture 4: The monotone sequence theorem in $\overline{\mathbb{R}}$. Series in $\overline{\mathbb{R}}$. Series with terms in $[0, \infty]$. Fact: series with non-negative terms can be rearranged arbitrarily and still give the same sum (finite or infinite). (Some special cases are proved in the printed notes. These results also follow from results on integration in Chapter 4.) Open sets in $\mathbb{R}$, defined as countable unions of open intervals.

Lecture 5: Closed sets in $\mathbb{R}$. Examples. Revision of pre-images for functions. Revision: a function f from $\mathbb{R}$ to $\mathbb{R}$ is continuous if and only if $f^{-1}(U)$ is open in $\mathbb{R}$ for every open subset U of $\mathbb{R}$ (and similarly for closed sets). Open/closed subsets of $\overline{\mathbb{R}}$ are discussed in the printed notes.

Chapter 2. Classes of sets Motivation for looking at different collections of sets (not all sets will be measurable). The power set of X, $\mathcal{P}(X)$ (or 2^X). Levels of abstraction, notation e.g $x \in \mathbb{R}$, $A \subseteq \mathbb{R}$, $\mathcal{F} \subseteq \mathcal{P}(X)$. Set operations revised: intersection, union and set difference. Countable intersections and unions. De Morgan's laws (finite and countable versions). Symmetric difference introduced, various equivalent definitions. Symmetric difference is associative (see question sheet 2).

Lecture 6: Semi-rings of sets: Intervals in $\mathbb{R}$, rectangles in $\mathbb{R}^2$. Half-open intervals $P = \{(a, b] : a, b \in \mathbb{R}, a \leq b\}$. Half-open rectangles. Rings of sets. Elementary figures in $\mathbb{R}$: finite (disjoint) unions of half-open intervals from P. Elementary figures in $\mathbb{R}^n$.

Lecture 7: Definition and examples of fields and σ-fields of subsets of a set X. Investigation of the properties of any σ-field $\mathcal{F}$ of subsets of $\mathbb{R}$ such that $\mathcal{F} \supseteq P$ (where P is as defined above). (Note that $\mathcal{P}(X)$ is such a σ-field.) Any such σ-field must include all open intervals, all open subsets of $\mathbb{R}$, all closed subsets of $\mathbb{R}$ and all countable subsets of $\mathbb{R}$ (including single-point sets $\{a\}$, where $a \in \mathbb{R}$, and also $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$). Question to be resolved later: are there any such σ-fields on $\mathbb{R}$ other than the σ-field $\mathcal{P}(\mathbb{R})$?

Lecture 8: The σ-field on X generated by a collection $\mathcal{C}$ of subsets of X, denoted in this module by $\mathcal{F}(\mathcal{C})$. This is the smallest possible σ-field on X which includes all of the sets in $\mathcal{C}$. More formally, $\mathcal{F}(\mathcal{C})$ is a σ-field on X, $\mathcal{C} \subseteq \mathcal{F}(\mathcal{C})$ and, whenever $\mathcal{G}$ is a σ-field on X such that $\mathcal{C} \subseteq \mathcal{G}$ then we have also $\mathcal{F}(\mathcal{C}) \subseteq \mathcal{G}$. Proof of the existence and properties of $\mathcal{F}(\mathcal{C})$. The σ-field, $\mathcal{B}$, of all Borel sets in $\mathbb{R}$ (also called Borel subsets of $\mathbb{R}$ or Borel measurable subsets of $\mathbb{R}$), $\mathcal{B}$: $\mathcal{B}$ is the σ-field generated by the collection of all open subsets of $\mathbb{R}$. Examples of Borel subsets of $\mathbb{R}$, including open sets, closed sets, intervals, countable sets, F_σ sets and G_δ sets. There are many other Borel sets. Brief comments on transfinite induction (beyond the scope of this module, but see books if interested).

Lecture 9: The Cantor middle-thirds set and the Cantor function. Short cuts for proving $\mathcal{F}(\mathcal{C}_1) = \mathcal{F}(\mathcal{C}_2)$. Proof that $\mathcal{F}(P) = \mathcal{B}$ (P and $\mathcal{B}$ as above).

Chapter 3. Measures and measure spaces Our aim is to measure the size of all 'sensible' subsets of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$ *etc.* (including at least all the Borel sets) in accordance with their length, area, or volume *etc.*. We will construct Lebesgue measure to do this. However some subsets of $\mathbb{R}^n$ will not fit into our collection of 'measurable' sets. An example of a 'non-measurable' subset of $[0, 1]$ (using equivalence classes modulo the rationals).

Lecture 10: Finished showing that the 'non-measurable' set constructed in the previous lecture cannot be assigned a length in a sensible way. Definition of (positive) measure on a collection $\mathcal{C}$ of subsets of X (with $\emptyset \in \mathcal{C}$). Simpler version of definition when $\mathcal{C} = \mathcal{F}$ is a σ-field. Stated the fact (proved later during the construction of Lebesgue measure) that length is a measure on our usual semi-ring P of half-open intervals in $\mathbb{R}$. This is not true for 'intervals of rationals', so some important property of the real numbers must be used. Examples of measures: the zero measure, the biggest possible measure and counting measure.

Lecture 11: Measures on rings. We will later see as part of general theory how to extend measure from a semi-ring to a ring. In particular, the ring of elementary figures in $\mathbb{R}$ (finite unions of sets in P) has a measure given by adding the lengths of the disjoint half-open intervals making up the set. Properties of measures on rings. Countable additivity (part of definition), finite additivity, monotonicity, countable subadditivity, continuity properties (the measure of a countable union is the limit of the measures of the finite unions, and the same goes for countable intersections provided the sets involved do not all have infinite measure).

Lecture 12: Problems with countable intersections when all the sets have infinite measure. Measurable spaces and measure spaces: counting measure on $\mathcal{P}(\mathbb{N})$, Lebesgue measure on the Borel sets in $\mathbb{R}$ (details of the construction of Lebesgue measure are given in Chapter 3 of the printed notes and will be covered later in the module). The Borel sets in $\overline{\mathbb{R}}$. Our default σ-fields on $\mathbb{R}$ and $\overline{\mathbb{R}}$ will in each case be the σ-field of Borel sets (on the relevant set). Integration in measure spaces is covered in Chapter 4. Integration on $\mathbb{N}$ with respect to counting measure is the same as summing series. Dominated convergence theorem stated in this setting in terms of limits of sequences of series under appropriate conditions. Properties that hold almost everywhere e.g. almost every real number is irrational (because $\lambda(\mathbb{Q}) = 0$, where λ is Lebesgue measure on $\mathbb{R}$).

Lecture 13: Definitions of Lebesgue outer measure, λ^*, and Lebesgue measure λ. Some standard properties stated (details to be proved later) for these. Restrictions of σ-fields to subsets (new σ-fields from old). Equivalence (almost everywhere equality) of functions on measure spaces. Positive measures and other kinds of measures: complex measures, real measures, signed measures. Hahn decomposition for complex/signed measures stated in the form $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ (where μ_i are positive measures, $1 \leq i \leq 4$). For Chapters 1-5 of this module we work only with positive measures, but the Hahn decomposition allows us to extend much of the theory effortlessly to more general measures. Finite (positive) measures and probability measures.

Lecture 14: *Chapter4: The Integral* Revision of Riemann integration: approximation of functions from below and above using step functions (staircase functions). The non-integrability of the characteristic function of $\mathbb{Q}$. The idea behind the Lebesgue integral: work with finite linear combinations of characteristic functions of sets more general than intervals (can use any measurable sets). These will be easy to integrate (in particular we will have no problem integrating $\chi_{\mathbb{Q}}$) and we will then extend our integral to cover *all* non-negative 'measurable' functions with values in $\overline{\mathbb{R}}$ (most functions you will ever meet are measurable!). Our theory of integration will then continue to discuss more general measurable functions with values in $\overline{\mathbb{R}}$. We will cover the theory in the setting of a general measure space, so our theory will apply to give us new information about values of series and Riemann integrals. Simple functions on X: definition (n.b. simple functions are real-valued), examples, standard form (using the distinct values, partition the set X and so form a finite linear combination of characteristic functions). Sums, products and linear combinations of simple functions are still simple functions. Every finite linear combination of characteristic functions is a simple function (even if the sets do not form a partition of X or the coefficients are not distinct).

Lecture 15: Comments on homework: every countable union of intervals is a Borel set but not every Borel set is a countable union of intervals. Continuous functions, images and pre-images revised. Measurable functions in terms of pre-images of measurable sets in the co-domain. Every continuous function from $\mathbb{R}$ to $\mathbb{R}$ is Borel measurable. Exercise: every monotone function from $\mathbb{R}$ to $\mathbb{R}$ is (Borel) measurable. Stated criteria for measurability of a function f from X to $\mathbb{R}$ or $\overline{\mathbb{R}}$ in terms of the measurability of the sets $\{x \in X : f(x) \leq a\}$.

Lecture 16: Functions defined on $\mathbb{N}$ are automatically measurable (when we use the σ-field most commonly used on $\mathbb{N}$, namely $\mathcal{P}(\mathbb{N})$). Proof of the criterion stated last time for measurability of functions (real-valued case). Several other equivalent criteria stated (left as exercises). Sketch of proof that every monotone function from $\mathbb{R}$ to $\mathbb{R}$ is (Borel) measurable. The function $-f$ is measurable if and only if f is measurable.

Lecture 17: The pointwise sup, inf, lim sup and lim inf of any sequence of measurable functions is measurable. Hence every pointwise limit of a sequence of measurable functions is a measurable function. A sum of two measurable functions is measurable provided that the sum is everywhere defined (see printed notes for proof). Simple measurable functions (measurable simple functions). Direct proof sketched that a sum of two simple measurable functions is again a simple measurable function.

Lecture 18: A product of two simple measurable functions is again a simple measurable function. Definition of the Lebesgue integral of non-negative, simple measurable functions and general non-negative measurable functions respect to a measure.

Tutorial session on the main three theorems of Chapter 4: the Monotone Convergence Theorem, Fatou's Lemma and the Dominated Convergence Theorem (discussed in the context of Riemann integrals of functions and sums of series). Students worked in groups to find counterexamples when conditions of the theorems are weakened, and an example where the inequality in Fatou's Lemma is strict. Answers were discussed.

Lecture 19: Monotone approximation from below of non-negative measurable functions using non-negative simple measurable functions. Deduction (from the corresponding result for simple measurable functions) of the fact that both the sum and the product of two non-negative measurable functions are measurable. Many results for general measurable functions can be deduced in the same way using the results for simple measurable functions and this method of approximation.

Lecture 20: Decomposition of $\overline{\mathbb{R}}$-valued functions into positive and negative parts: $f = f^+ - f^-$. The function f is measurable if and only if both f^+ and f^- are. The pointwise maximum of two $\overline{\mathbb{R}}$-valued measurable functions is a measurable function. Recalled the definition of the Lebesgue integral of non-negative, simple measurable functions and general non-negative measurable functions over a set E respect to a measure μ (notation: $I_E(s,\mu)$ [non-standard] for non-negative, simple measurable functions and $\int_E f \mathrm{d}\mu$ for general non-negative measurable functions). Defined (where possible) the integral of a measurable $\overline{\mathbb{R}}$-valued function f to be $\int_E f \mathrm{d}\mu = \int_E f^+ \mathrm{d}\mu - \int_E f^- \mathrm{d}\mu$. Some elementary facts about integrals of simple functions stated (intuitively obvious, proofs in printed notes or on question sheet 4, some proofs discussed in lectures).

Lecture 21: Further elementary facts about integrals of non-negative simple measurable functions. The integral of a sum of two such functions is the sum of the integrals. When s is non-negative, simple measurable, then the function $\phi(E) = I_E(s,\mu)$ is a measure on $\mathcal{F}$. Recalled definition of the Lebesgue integral of a non-negative measurable function, $\int_E f \mathrm{d}\mu$. In particular this gives the same value as before for simple measurable functions: $I_E(s,\mu) = \int_E s\mathrm{d}\mu$. Thus we may safely switch to the new notation, but maintain our old results. For example, when s is non-negative, simple measurable, then the function $\phi(E) = \int_E s\mathrm{d}\mu$ is a measure on $\mathcal{F}$. Elementary facts about integrals of non-negative, measurable functions stated (proofs in printed notes, most follow directly from the definitions and the results for non-negative, simple measurable functions).

Lecture 22: Further elementary facts about integrals of non-negative, measurable functions discussed (proofs in printed notes). Revision of continuity properties of measures (nested unions). Statement and proof of the Monotone Convergence Theorem. Typical application: deduction of less elementary facts about integrals of non-negative, measurable functions using facts about simple measurable functions and monotone approximation. Integral of a sum of two non-negative, measurable functions:

$$\int_X (f+g)\mathrm{d}\mu = \int_X f \ \mathrm{d}\mu + \int_X g \ \mathrm{d}\mu.$$

Lecture 23: Further corollaries of the MCT: for a non-negative measurable function f and $\alpha \in [0,\infty)$ we have

$$\int_X \alpha f \mathrm{d}\mu = \alpha \int_X f \ \mathrm{d}\mu$$

(this may also be proved by elementary means). For non-negative measurable functions f_n,

$$\int_X \left(\sum_{n=1}^{\infty} f_n \right) \ \mathrm{d}\mu = \sum_{n=1}^{\infty} \left(\int_X f_n \ \mathrm{d}\mu \right).$$

In particular (using counting measure on $\mathbb{N}$), for non-negative extended real numbers $a_{n,k}$,

$$\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} a_{n,k} \right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{n,k} \right).$$

For any non-negative measurable function f, the function $\Phi(E) = \int_E f \ \mathrm{d}\mu$ is a measure on $\mathcal{F}$. Statement and proof of Fatou's Lemma.

Lecture 24: Recalled definition of integral (where defined) of a measurable, $\overline{\mathbb{R}}$-valued function f. The (measurable) function f is integrable (or summable) on E if both f^+ and f^- have finite integral on E. The set of integrable functions f such that f take values in $\mathbb{R}$ (non-standard) is denoted by $L^1(\mu)$. (Some authors write $L^1(X, \mu)$ or $L^1(X, \mathrm{d}\mu)$, and usually they allow f to be $\overline{\mathbb{R}}$-valued, but this makes no real difference to the theory.) $L^1(\mu)$ is a vector space of functions on X, and, for f, g in $L^1(\mu)$ and α, β in $\mathbb{R}$, we have

$$\int_X (\alpha f + \beta g)\ \mathrm{d}\mu = \alpha \int_X f\ \mathrm{d}\mu + \beta \int_X g\ \mathrm{d}\mu.$$

Lecture 25: Recalled Fatou's Lemma and discussed the inequality

$$\left| \int_X f\ \mathrm{d}\mu \right| \leq \int_X |f|\ \mathrm{d}\mu$$

(valid whenever f is a measurable $\overline{\mathbb{R}}$-valued function such that at least one of $\displaystyle\int_X f^+\ \mathrm{d}\mu$ and $\displaystyle\int_X f^-\ \mathrm{d}\mu$ is finite). All our convergence theorems are also valid for $\displaystyle\int_E$ rather than $\displaystyle\int_X$, where $E \in \mathcal{F}$ (using question sheet 4, question 1). These convergence theorems also remain true when the given conditions hold almost everywhere with respect to μ rather than for all x. Statement and proof of the Dominated Convergence Theorem (DCT).

Lecture 26: Further comments on conditions that hold almost everywhere: since a countable union of sets of measure zero is always another set with measure zero, you can always throw out any countable collection of 'bad' sets which have measure zero (where conditions fail) and work on the remainder of the space (where the integrals are the same as over the whole space). The connection between the Riemann integral and the Lebesgue integral (with respect to Lebesgue measure on an interval): these agree for all Riemann integrable functions (proof sketched, based on approximation of a Riemann integrable function by step functions). This allows us to use the notation $\displaystyle\int_a^b f(x)\ \mathrm{d}x$ for the Lebesgue integral $\displaystyle\int_{[a,b]} f\ \mathrm{d}\lambda$ of a Lebesgue integrable function (even if it is not Riemann integrable). We may also use the notation $\displaystyle\int_X f(x, y)\ \mathrm{d}\mu(x)$ when integrating functions of more than one variable.

Lecture 27: Recalled definition of Lebesgue outer measure λ^*. Explained our strategy for construction Lebesgue measure: prove that length is a measure on P, then show that every measure on a semi-ring $\mathcal{S}$ can be extended to a complete measure on a σ-field $\mathcal{F} \supseteq \mathcal{S}$ (using the theory of outer measures). In particular, using Lebesgue outer measure allows us to extend our notion of length (defined on P) to a measure on the Borel sets (and more). Statements and proofs of some elementary facts about finite unions of half-open intervals and their lengths. Brief discussion of compactness: recalled Heine-Borel Theorem and looked at some open covers of $(0, 3)$ and $[0, 3)$ (which are NOT compact). Compactness of $[a, b]$: follows from the Heine-Borel Theorem, or see standard proof in the printed notes, or can deduce the result from a Lemma concerning the 'Lebesgue number' for a covering. We only need the result concerning the situation when $[a, b]$ is a subset of a union of a sequence of open intervals.

Lecture 28: Proof of Lemma in the specific situation above (compactness result for closed intervals follows). Connection with length: if $[a, b]$ is a subset of a union of a sequence of open intervals, then the sum of the lengths of the open intervals must be at least $b - a$. (This is NOT true for the set $[a, b] \cap \mathbb{Q}$.) Length of half-open intervals, $\mu((a, b]) = b - a$, really is a measure on our usual semi-ring P of half-open intervals.

Lecture 29: Extension of a measure from a semi-ring to a ring: formula given, details in printed notes. In particular, 'total length' of a finite disjoint union of half open intervals gives a measure on the ring generated by our usual semi-ring P. Definition and examples of outer measures. Standard procedure for obtaining an outer measure from a measure on a semi-ring (proof that this gives an outer measure omitted: see printed notes). In particular we will work with Lebesgue outer measure, which is obtained in this standard way from our usual measure on P. Stated that such an outer measure takes the correct values on sets in the semi-ring (proof later). Defined the notion of measurable with respect to an outer measure (μ^*-measurable sets). Elementary properties of measurability.

Lecture 30: Tutorial session on measures and outer measures.

Lecture 31: Recalled definition of μ^*-measurable sets. The collection of μ^*-measurable sets is a σ-field and the restriction of μ^* to this σ-field is a measure.

32: Extension of measures from semi-rings to σ-fields using outer measures. In particular this gives us Lebesgue measure on the Lebesgue measurable subsets of $\mathbb{R}$ extending our notion of length of half-open intervals (defined on P). Some properties of Lebesgue measure: translation invariance, regularity.