Section 5: Outer measures and the construction of Lebesgue measure

We begin our construction of Lebesgue measure by checking that the notion of length we are used to really does give a measure on our usual semi-ring of half-open intervals, P. In order to do this we need to briefly discuss *compactness*.

Compactness of Closed Intervals

In terms of sequences, a metric space is *compact* if every sequence in X has a subsequence which converges in X. By the Bolzano-Weierstrass theorem, then, every closed and bounded interval $[a,b]$ in $\mathbb{R}$ is compact. However there is a topological version of compactness which we need here concerning coverings using open sets.

Let $a < b \in \mathbb{R}$. Suppose we have a collection of open sets in $\mathbb{R}$, $\{U_\gamma : \gamma \in \Gamma\}$ s.t.

$$[a,b] \subseteq \bigcup_{\gamma \in \Gamma} U_\gamma .$$

Then it is always true that there are

$$\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$$

s.t.

$$[a,b] \subseteq \bigcup_{j=1}^{n} U_{\gamma_j} .$$

We shall only need a special case of this fact, (see Proposition 5.2 below). See books on metric and topological spaces for more details on compactness.

Lemma 5.1

(i) If

$$(a,b] \subseteq \bigcup_{k=1}^{n} (a_k, b_k],$$

then

$$\sum_{k=1}^{n} (b_k - a_k) \geqslant b - a.$$

(ii) If

$$(a,b] \supseteq \dot{\bigcup_{k=1}^{n}} (a_k, b_k]$$

then

$$\sum_{k=1}^{n} (b_k - a_k) \leqslant b - a.$$

(iii) If

$$(a,b] = \dot{\bigcup_{k=1}^{n}} (a_k, b_k]$$

then

$$(b-a) = \sum_{k=1}^{n} (b_k - a_k).$$

Proof (There is a fairly easy direct proof available but we use the Riemann integral.)

We can assume that all the intervals

$$(a_k, b_k], \qquad (a,b]$$

are contained in some closed interval $[-m, m]$. Then

$$(b-a) = \int_{-m}^{m} \chi_{(a,b]}(t)\ \mathrm{d}t$$

and

$$(b_k - a_k) = \int_{-m}^{m} \chi_{(b_k, a_k]}(t)\ \mathrm{d}t.$$

(i) If

$$(a,b] \subseteq \bigcup_{k=1}^{n} (a_k, b_k]$$

then

$$\chi_{(a,b]} \leqslant \sum_{k=1}^{n} \chi_{(a_k, b_k]}$$

at every point, so

$$\begin{aligned} b-a &= \int_{-m}^{m} \chi_{(a,b]}(t)\ \mathrm{d}t \\ &\leqslant \int_{-m}^{m} \sum_{k=1}^{n} \chi_{(a_k, b_k)}(t)\ \mathrm{d}t \\ &= \sum_{k=1}^{n} (b_k - a_k). \end{aligned}$$

(ii) If

$$(a,b] \geqslant \dot{\bigcup_{k=1}^{n}} (a_k, b_k)$$

then

$$\chi_{(a,b]}(t) \geqslant \sum_{k=1}^{n} \chi_{(a_k,b_k)}(t) \quad \forall t.$$

Integrating as in (i),

$$(b-a) \geqslant \sum_{k=1}^{n} (b_k - a_k).$$

(iii) If

$$(a,b] = \dot{\bigcup_{k=1}^{n}} (a_k, b_k],$$

then (i) and (ii) apply and

$$(b-a) = \sum_{k=1}^{n} (b_k - a_k). \qquad \square$$

A similar result holds in n-dimensions (see question sheet 2 for $\mathbb{R}^2$).

Recall: closed intervals $[a,b]$ are compact. In particular, we have the following.

Proposition 5.2

Let $[a,b] \subseteq \mathbb{R}$ be a closed interval, and suppose that

$$(a_k, b_k) \quad (k = 1, 2, 3, \ldots)$$

are open intervals s.t.

$$[a,b] \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k).$$

Then $\exists\, m \in \mathbb{N}$ s.t.

$$[a,b] \subseteq \bigcup_{k=1}^{m} (a_k, b_k). \qquad \square$$

Returning to half-open intervals, we need the following.

Corollary 5.3

If

$$[a,b] \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k)$$

then

$$(b-a) \leqslant \sum_{k=1}^{\infty} (b_k - a_k).$$

Proof

By Proposition 5.2 there exists $m \in \mathbb{N}$ such that $[a, b] \subseteq \bigcup_{k=1}^{m} (a_k, b_k)$. But then $(a, b] \subseteq \bigcup_{k=1}^{m} (a_k, b_k]$, and the result follows from 5.1(i). □

Theorem 5.4

Define $\mu \colon P \to [0, \infty]$ by $\mu((a, b]) = b - a$. Then μ is a measure on P.

Proof

Certainly $\mu(\varnothing) = 0$. Now suppose that

$$(a, b] = \dot{\bigcup_{n=1}^{\infty}} (a_n, b_n].$$

We must show that

$$b - a = \sum_{n=1}^{\infty} (b_n - a_n).$$

Certainly we may assume $b > a$. First note that, for all $m \in \mathbb{N}$,

$$\dot{\bigcup_{n=1}^{m}} (a_n, b_n] \subseteq (a, b]$$

and so, by 5.1(ii),

$$\sum_{n=1}^{m} (b_n - a_n) \leqslant b - a.$$

Since this is true for all $m \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} (b_n - a_n) \leqslant b - a.$$

To conclude the proof we must prove the reverse inequality. Let $\varepsilon > 0$. Then, provided that $\varepsilon < b - a$,

$$[a + \varepsilon, b] \subseteq \bigcup_{n=1}^{\infty} \left(a_n, b_n + \frac{\varepsilon}{2^n} \right)$$

so, by Corollary 5.3,

$$(b - (a + \varepsilon)) \leqslant \sum_{n=1}^{\infty} \left(b_n + \frac{\varepsilon}{2^n} - a_n \right),$$

i.e.

$$b - a - \varepsilon \leqslant \left(\sum_{n=1}^{\infty} (b_n - a_n) \right) + \varepsilon,$$

$$b - a \leqslant \sum_{n=1}^{\infty} (b_n - a_n) + 2\varepsilon.$$

Since this is true for all $\varepsilon \in (0, b-a)$, we obtain

$$(b-a) \leqslant \sum_{n=1}^{\infty} (b_n - a_n)$$

as required. □

A similar result holds in $\mathbb{R}^n$: e.g. in $\mathbb{R}^2$, the function ν defined on half-open rectangles by

$$\nu((a,b]\times(c,d]) = (b-a)(d-c)$$

is also a measure (on P^2).

We now wish to measure the size of more complicated sets. A good start is to extend μ to a measure on the elementary figures $\mathscr{E}$ (finite unions of sets in P). The fact that this is possible is a special case of a more general theorem. First we define extension.

Definition 5.5

Let X be a set, and let $\mathscr{C}_1, \mathscr{C}_2$ be subsets of $\mathscr{P}(X)$ with $\mathscr{C}_1 \subseteq \mathscr{C}_2$. Let

$$\mu_1 \colon \mathscr{C}_1 \to [0,\infty], \qquad \mu_2 \colon \mathscr{C}_2 \to [0,\infty]$$

be *functions*. Then μ_2 is an *extension* of μ_1 if

$$\mu_2(E) = \mu_1(E) \quad (E \in \mathscr{C}_1).$$

In this case we say μ_1 is the *restriction* of μ_2 to $\mathscr{C}_1$.

[This agrees with the usual notions of extension and restriction for functions.]

Note on summation

In the next theorem we will need to be able to change the order of summation in various series. Recall, when we proved Proposition 1.9 we saw that if

$$a_{jk} \in [0,\infty], \qquad (j = 1,2,3,\ldots, \quad 1 \leqslant k \leqslant m)$$

then

$$\sum_{j=1}^{\infty}\left(\sum_{k=1}^{m} a_{jk}\right) = \sum_{k=1}^{m}\left(\sum_{j=1}^{\infty} a_{jk}\right).$$

We shall now show how to extend measures from semi-rings to rings, and in particular from P to $\mathscr{E}$.

More useful facts about series of elements of $[0,\infty]$

Recall Proposition 1.10.

Suppose we have $a_{ik} \in [0, \infty]$, where $i \in \mathbb{N}$, and $1 \leqslant k \leqslant n_i$. Then the set

$$\{(i, k) \colon k, i \in \mathbb{N},\ 1 \leqslant k \leqslant n_i\}$$

is countable, so we can enumerate this set as

$$\{(i_t, k_t) \colon t = 1, 2, 3, \ldots\}.$$

$$\sum_{i=1}^{\infty} \left(\sum_{k=1}^{n_i} a_{ik} \right) = \sum_{t=1}^{\infty} a_{i_t k_t}. \qquad (*)$$

From this we obtain a crucial fact about measures.

Lemma 5.6

Let X be a set, let $\mathscr{C} \subseteq \mathscr{P}(X)$, suppose $\phi \in \mathscr{C}$, and let $\mu \colon \mathscr{C} \to [0, \infty]$ be a measure.

If $A \in \mathscr{C}$, and A satisfies

$$A = \dot{\bigcup_{i=1}^{\infty}} \dot{\bigcup_{k=1}^{n_i}} D_{ik},$$

where $n_1, n_2, \ldots \in \mathbb{N}$, and the sets D_{ik} are all in $\mathscr{C}$, then

$$\mu(A) = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \mu(D_{ik}).$$

[In some books this is regarded as obvious!]

Proof

Let $(i_1, k_1), (i_2, k_2), (i_3, k_3), \ldots$ be an enumeration of the set

$$\{(i, k) \colon k, i \in \mathbb{N},\ 1 \leqslant k \leqslant n_i\}.$$

Then

$$A = \dot{\bigcup_{t=1}^{\infty}} D_{i_t k_t}$$

so

$$\begin{aligned} \mu(A) &= \sum_{t=1}^{\infty} \mu(D_{i_t k_t}) \\ &= \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \mu(D_{ik}) \end{aligned}$$

by $(*)$ with $a_{ik} = \mu(D_{ik})$.

With these facts at our disposal, we can prove our theorem.

Theorem 5.7

Let $\mathscr{S}$ be a semi-ring of subsets of a set X, and let $\mu_1\colon \mathscr{S} \to [0,\infty]$ be a measure. Then there is a unique measure $\mu_2\colon R(\mathscr{S}) \to [0,\infty]$ such that μ_2 is an extension of μ_1.

Proof

For any $A \in R(\mathscr{S})$ there are disjoint sets $A_1,\ldots,A_m$ in $\mathscr{S}$ with

$$A = \dot{\bigcup_{j=1}^{m}} A_j.$$

The only possible value for $\mu_2(A)$ is

$$\sum_{j=1}^{m} \mu_1(A_j).$$

Thus if such a measure μ_2 exists it is unique. We now wish to *define*

$$\mu_2\left(\dot{\bigcup_{j=1}^{m}} A_j\right) = \sum_{j=1}^{m} \mu_1(A_j)$$

whenever $A_1,\ldots,A_m$ are disjoint sets in $\mathscr{S}$. To see that μ_2 is well defined, suppose that

$$\dot{\bigcup_{j=1}^{m}} A_j = \dot{\bigcup_{k=1}^{n}} B_k,$$

with A_j, B_k all in $\mathscr{S}$. Then set $C_{jk} = A_j \cap B_k$, and obtain

$$A_j = \dot{\bigcup_{k=1}^{n}} C_{jk} \qquad \text{all } j,$$

$$B_k = \dot{\bigcup_{j=1}^{m}} C_{jk} \qquad \text{all } k.$$

Thus

$$\begin{aligned} \sum_{j=1}^{m} \mu_1(A_j) &= \sum_{j=1}^{m} \sum_{k=1}^{n} \mu_1(C_{jk}) \\ &= \sum_{k=1}^{n} \sum_{j=1}^{m} \mu_1(C_{jk}) \\ &= \sum_{k=1}^{n} \mu_1(B_k). \end{aligned}$$

Thus μ_2 is well defined. To see that μ_2 is a measure, suppose that

$$A \in R(\mathscr{S}) \qquad \text{and} \qquad A = \dot{\bigcup_{i=1}^{\infty}} B_i$$

with each $B_i \in R(\mathscr{S})$. Then there are pairwise disjoint sets A_j in $\mathscr{S}$ $(1 \leqslant j \leqslant m)$ with

$$A = \dot{\bigcup_{j=1}^{m}} A_j.$$

Also, for each i, there are disjoint sets $C_{i1}, C_{i2}, \ldots, C_{in_i}$ in $\mathscr{S}$ such that

$$B_i = \dot{\bigcup_{k=1}^{n_i}} C_{ik}.$$

We have

$$A_j = \dot{\bigcup_{i=1}^{\infty}} (A_j \cap B_i) = \dot{\bigcup_{i=1}^{\infty}} \dot{\bigcup_{k=1}^{n_i}} (A_j \cap C_{ik})$$

and
$$B_i = \dot{\bigcup_{j=1}^{m}} \dot{\bigcup_{k=1}^{n_i}} (A_j \cap C_{ik}).$$

Since the sets $(A_j \cap C_{ik})$ are in $\mathscr{S}$ we have

$$\mu_2(B_i) = \sum_{j=1}^{m} \sum_{k=1}^{n_i} \mu_1(A_j \cap C_{ik})$$

and, since μ_1 is a measure, setting $D_{ik} = A_j \cap C_{ik}$ in Lemma 5.6 gives

$$\mu_1(A_j) = \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \mu_1(A_j \cap C_{ik}).$$

Thus

$$\begin{aligned}\mu_2(A) = \sum_{j=1}^{m} \mu_1(A_j) &= \sum_{j=1}^{m} \sum_{i=1}^{\infty} \sum_{k=1}^{n_i} \mu_1(A_j \cap C_{ik}) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{m} \sum_{k=1}^{n_i} \mu_1(A_j \cap C_{ik}) \\ &= \sum_{i=1}^{\infty} \mu_2(B_i).\end{aligned}$$

This concludes the proof. □

In particular, the measure on P

$$\mu((a, b]) = b - a$$

extends to a unique measure on the ring of elementary figures in $\mathbb{R}$.

Example 5.8

Define $\mu_1 \colon P \to [0, \infty]$ by $\mu_1((a, b]) = b - a$. Then μ_1 has a unique extension $\mu_2 \colon \mathscr{E} \to [0, \infty]$ which is a measure satisfying

$$\mu_2\left(\dot{\bigcup_{k=1}^{n}} (a_k, b_k]\right) = \sum_{k=1}^{n} (b_k - a_k).$$

(Similarly in higher dimensions.)

For our main extension theorem we will need the notion of an outer measure μ^* and sets which are ‘measurable with respect to μ^*’.

Definition 5.9

Let X be a set, and let

$$\mu^*\colon \mathscr{P}(X) \to [0,\infty].$$

Then μ^* is an *outer measure* on X if

(i) $\mu^*(\phi) = 0$,

(ii) if $A, B \in \mathscr{P}(X)$ and $A \subseteq B$ then $\mu^*(A) \leqslant \mu^*(B)$ (μ^* is *monotone*),

(iii) if $A \subseteq \bigcup_{n=1}^{\infty} A_n$, where A, A_n are subsets of X, then $\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu^*(A_n)$. ($\mu^*$ is *countably subadditive*.)

Note

An outer measure on X is defined on *all* subsets of X.

Examples

(i) Any measure on $\mathscr{P}(X)$ is also an outer measure on X.

(ii) Defining

$$\mu^*(E) = \begin{cases} 0 & (E = \varnothing), \\ 1 & (E \neq \varnothing), \end{cases}$$

defines an outer measure which is not a measure (provided that X has at least two points!).

Definition 5.10

Let X be a set and let μ^* be an outer measure on X. A set $A \subseteq X$ is said to be *measurable with respect to* μ^* (or μ^**-measurable*) if, for *every* set $E \subseteq X$, the equality

$$\mu^*(E) = \mu^*(E \setminus A) + \mu^*(E \cap A)$$

holds.

Notes

(i) We know $\mu^*(E) \leqslant \mu^*(E \setminus A) + \mu^*(E \cap A)$ by subadditivity, thus A is μ^*-measurable if and only if, for all $E \subseteq X$,

$$\mu^*(E) \geqslant \mu^*(E \setminus A) + \mu^*(E \cap A).$$

(ii) Similarly, if $\mu^*(E) = \infty$ the equality is automatic, so the condition need only be checked for those E with $\mu^*(E) < \infty$.

(iii) A is μ^*-measurable if and only if $X \setminus A$ is μ^*-measurable, because

$$E \setminus A = E \cap (X \setminus A), \quad \text{while}$$

$$E \cap A = E \setminus (X \setminus A).$$

We will be interested in using a measure defined on a ring to define an outer measure on all subsets of a set.

[e.g. we shall show that defining, for $A \subseteq \mathbb{R}$,

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^{\infty} (b_n - a_n) \colon a_n \leqslant b_n \in \mathbb{R},\ A \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \right\}$$

μ^* is an outer measure on $\mathbb{R}$. We will show that all the Borel sets are μ^*-measurable, and that the restriction of μ^* to the Borel sets is a measure.]

We now begin to prove the results we need about outer measures. The first Lemma is fairly weak.

Lemma 5.11

Let X be a set and let $\mu^*\colon \mathcal{P}(X) \to [0, \infty]$ be an outer measure. Set

$$\mathcal{F} = \{A \in \mathcal{P}(X) \colon A \text{ is } \mu^*\text{-measurable}\}.$$

Then $\mathcal{F}$ is a field, and, for *any* $E \subseteq X$, and any disjoint sets $A_1, A_2, \ldots, A_n \in \mathcal{F}$,

$$\mu^*\left(\dot{\bigcup_{k=1}^{n}} (E \cap A_k) \right) = \sum_{k=1}^{n} \mu^*(E \cap A_k).$$

In particular, μ^* is finitely additive on $\mathcal{F}$.

Proof

Let $A, B \in \mathcal{F}$, and let $E \subseteq X$. Then

note that $$E \setminus (A \cup B) = (E \setminus A) \setminus B,$$

and that $$E \cap (A \cup B) = ((E \setminus A) \cap B) \cup (E \cap A).$$

Thus

$$\begin{aligned} \mu^*(E) &\leqslant \mu^*(E \setminus (A \cup B)) + \mu^*(E \cap (A \cup B)) \\ &= \mu^*((E \setminus A) \setminus B) + \mu^*(((E \setminus A) \cap B) \cup (E \cap A)) \\ &\leqslant \mu^*((E \setminus A) \setminus B) + \mu^*((E \setminus A) \cap B) + \mu^*(E \cap A) \end{aligned}$$

(by subadditivity)

$$= \mu^*(E \setminus A) + \mu^*(E \cap A)$$

(B is measurable)

$$= \mu^*(E)$$

(A is measurable).

Since $\mu^*(E)$ appears at both ends, all the inequalities in the middle are equalities, in particular,

$$\mu^*(E) = \mu^*(E \setminus (A \cup B)) + \mu^*(E \cap (A \cup B)).$$

This shows $A \cup B$ is measurable.

The fact that $\varnothing$ is measurable is trivial, and we already know that

$$A \in \mathcal{F} \Rightarrow X \setminus A \in \mathcal{F}.$$

Thus $\mathcal{F}$ is a field.

For the last part, let $E \subseteq X$ and define

$$\nu \colon \mathcal{F} \to [0, \infty] \quad \text{by}$$

$$\nu(A) = \mu^*(E \cap A).$$

Then we are required to prove that ν is finitely additive on $\mathcal{F}$.

[NOTE: If $A_1, \ldots, A_n$ are disjoint sets in $\mathcal{F}$, then

$$\dot{\bigcup_{k=1}^{n}} (E \cap A_k) = E \cap \dot{\bigcup_{k=1}^{n}} A_k.$$]

It is enough to show that if $A, B \in \mathcal{F}$ and $A \cap B = \varnothing$ then $\nu(A \cup B) = \nu(A) + \nu(B)$. But

$$\begin{aligned} \nu(A \cup B) &= \mu^*(E \cap (A \cup B)) \\ &= \mu^*((E \cap (A \cup B)) \cap A) + \mu^*((E \cap (A \cup B)) \setminus A) \end{aligned}$$

(since A is measurable)

$$= \mu^*(E \cap A) + \mu^*(E \cap B)$$

(since $A \cap B = \varnothing$)

$$= \nu(A) + \nu(B)$$

as claimed.

The result follows. □

We shall see that, in fact, $\mathcal{F}$ is a σ-field, and μ^* restricted to $\mathcal{F}$ is a measure. We will then be able to prove the extension results we want.

Lemma 5.12

If $\mathcal{F}$ is a field of subsets of a set X, then the following are equivalent:

(i) $\mathcal{F}$ is a σ-field,

(ii) whenever $A_1, A_2, A_3, \ldots$ are *pairwise disjoint* sets in $\mathcal{F}$ then

$$\dot{\bigcup_{n=1}^{\infty}} A_n \in \mathcal{F}.$$

Proof

(i) $\Rightarrow$ (ii) is trivial,
(ii) $\Rightarrow$ (i) assume (ii) holds.
Let $B_1, B_2, B_3, \ldots \in \mathscr{F}$.

We must show $\bigcup_{n=1}^{\infty} B_n \in \mathscr{F}$, to see this set

$$A_1 = B_1, \quad A_{n+1} = B_{n+1} \setminus \bigcup_{k=1}^{n} B_k \quad (n = 1, 2, 3, \ldots).$$

Then $A_1, A_2, A_3, \ldots$ are pairwise disjoint elements of $\mathscr{F}$.

Also

$$\bigcup_{n=1}^{\infty} B_n = \dot{\bigcup_{n=1}^{\infty}} A_n$$

and this is in $\mathscr{F}$ by (ii). □

Recall

If X is a non-empty set and μ^* is an outer measure on X, then, defining $\mathscr{F}$ to be the set of μ^* – measurable subsets of X we know that $\mathscr{F}$ is a field. Also, for any $E \subseteq X$ and sets $A_1, A_2, \ldots, A_n \in \mathscr{F}$ which are pairwise disjoint.

We have

$$\mu^*\left(\dot{\bigcup_{k=1}^{n}} E \cap A_k\right) = \sum_{k=1}^{n} \mu^*(E \cap A_k). \tag{*}$$

[Taking $E = X$, (*) shows μ^* is finitely additive on $\mathscr{F}$.]

Theorem 5.13

Let μ^* be an outer measure on a non-empty set X. Let $\mathscr{F}$ be the set of μ^*-measurable subsets of X. Then

(i) $\mathscr{F}$ is a σ-field on X,

(ii) for any set $E \subseteq X$ and pairwise disjoint sets $A_1, A_2, \ldots$ in $\mathscr{F}$,

$$\mu^*\left(\bigcup_{n=1}^{\infty} E \cap A_n\right) = \sum_{n=1}^{\infty} \mu^*(E \cap A_n),$$

(iii) the restriction of μ^* to $\mathscr{F}$ is a measure.

Proof

First note that (iii) follows immediately from (ii) by setting $E = X$, so we need only prove (i) and (ii). Also we know $\mathscr{F}$ is a field, so to prove (i) we need only check countable disjoint unions.

To prove (i) and (ii), let $A_1, A_2, \ldots$ be pairwise disjoint elements of $\mathcal{F}$ and let $E \subseteq X$.

Set

$$A = \dot{\bigcup_{n=1}^{\infty}} A_n.$$

Note that

$$E \cap A = \dot{\bigcup_{n=1}^{\infty}} (E \cap A_n).$$

Since μ^* is countably subadditive, we have

$$\mu^*(E \cap A) = \mu^*\left(\bigcup_{n=1}^{\infty} E \cap A_n\right) \leqslant \sum_{n=1}^{\infty} \mu^*(E \cap A_n). \tag{1}$$

Set

$$B_n = \bigcup_{k=1}^{n} A_n.$$

We have $B_n \in \mathcal{F}$ and $B_n \subseteq A$. Thus

$$\mu^*(B_n) \leqslant \mu^*(A)$$

(by monotocity) and

$$\mu^*(B_n \cap E) \leqslant \mu^*(E \cap A).$$

By Lemma 5.11,

$$\begin{aligned} \mu^*(B_n \cap E) &= \mu^*\left(\bigcup_{k=1}^{n} E \cap A_k\right) \\ &= \sum_{k=1}^{n} \mu^*(E \cap A_k). \end{aligned}$$

Thus

$$\mu^*(A \cap E) \geqslant \sum_{k=1}^{n} \mu^*(E \cap A_k).$$

Letting $n \to \infty$, we have

$$\mu^*(A \cap E) \geqslant \sum_{k=1}^{\infty} \mu^*(E \cap A). \tag{2}$$

We thus obtain (using (1) and (2))

$$\mu^*(E \cap A) = \sum_{k=1}^{\infty} \mu^*(E \cap A_k),$$

proving (ii).

To show that

$$\dot{\bigcup_{n=1}^{\infty}} A_n \in \mathscr{F}$$

we need to show that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

We know that $B_n \in \mathscr{F}$, so we have

$$\begin{aligned} \mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \setminus B_n) \\ &\geqslant \mu^*(E \cap B_n) + \mu^*(E \setminus A) \end{aligned}$$

(since $E \setminus B_n \supseteq E \setminus A$, and μ^* is monotone)

$$= \left(\sum_{k=1}^{n} \mu^*(E \cap A_k) \right) + \mu^*(E \setminus A).$$

Letting $n \to \infty$ we have

$$\begin{aligned} \mu^*(E) &\geqslant \left(\sum_{k=1}^{\infty} \mu^*(E \cap A_k) \right) + \mu^*(E \setminus A) \\ &= \mu^*(E \cap A) + \mu^*(E \setminus A) \end{aligned}$$

(by (ii))

$$\geqslant \mu^*(E).$$

Thus equality holds, and

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A).$$

This is true for all $E \subseteq X$, so A is measurable. □

Recall

Let X be a set, and define

$$\mu^*(E) = \begin{cases} 0 & E = \varnothing, \\ 1 & \text{otherwise.} \end{cases}$$

Then μ^* is an outer measure. The theorem applies to tell us the collection of μ^* measurable sets is a σ-field on which μ^* is a measure.

Exercise: Show that only $\varnothing$ and X are μ^*-measurable.

We now state without proof another fact about double series.

Proposition 5.14

If $a_{nk} \in [0, \infty]$ for $n, k \in \mathbb{N}$ then whatever order we add up the extended real numbers a_{nk} we always get the same answer: in particular

if you enumerate $\mathbb{N} \times \mathbb{N}$ as (n_1, k_1), (n_2, k_2), $(n_3, k_3), \ldots$ then

$$\sum_{j=1}^{\infty} a_{n_j k_j} = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{\infty} a_{nk} \right). \qquad \square$$

Remark: all these results about series have elementary proofs, but can also be deduced from results in our section on integration (later).

Lemma 5.15

Let X be a set, let $\mathscr{C}$ be a set of subsets of X with $\varnothing \in \mathscr{C}$. Suppose that $\mu: \mathscr{C} \to [0, \infty]$ is a function, and that $\mu(\varnothing) = 0$.

Define

$$\mu^*(A) = \inf\left\{ \sum_{n=1}^{\infty} \mu(E_n) \colon E_1, E_2 \ldots \in \mathscr{C} \text{ and } A \subseteq \bigcup_{n=1}^{\infty} E_n \right\}.$$

Then μ^* is an outer measure on X.

Remarks

We will use Proposition 5.14 in this proof. Note that $\mu^*(A)$ may be $+\infty$ for two reasons:

either there are no sets $E_1, E_2, E_3, \ldots$ in $\mathscr{C}$ with

$$A \subseteq \bigcup_{n=1}^{\infty} E_n,$$

in which case $\mu^*(A) = \inf(\varnothing) = +\infty$ or it could be that

$$\mu^*(A) = \inf\{\infty\}.$$

But certainly

$$\mu^*(A) \in [0, \infty] \quad \forall\, A \subseteq X.$$

Proof

Certainly $\mu^*: \mathscr{P}(X) \to [0, \infty]$. To see that $\mu^*(\varnothing) = 0$ take

$$E_1 = E_2 = \ldots = \varnothing \in \mathscr{C}.$$

Then

$$\varnothing \subseteq \bigcup_{n=1}^{\infty} E_n,$$

so

$$\mu^*(\varnothing) \leqslant \sum_{n=1}^{\infty} \mu(E_n) = 0.$$

Thus $\mu^*(\varnothing) = 0$.

To check monotonicity, suppose that $A \subseteq F$. Set

$$S_A = \left\{ \sum_{n=1}^{\infty} \mu(E_n) \colon E_1, \ldots \in \mathscr{C},\ A \subseteq \bigcup_{n=1}^{\infty} E_n \right\}$$
$$S_F = \left\{ \sum_{n=1}^{\infty} \mu(E_n) \colon E_1, \ldots \in \mathscr{C},\ F \subseteq \bigcup_{n=1}^{\infty} E_n \right\}$$

since $A \subseteq F$, we have $S_F \subseteq S_A$.

$$\therefore \qquad \inf(S_A) \leqslant \inf(S_F),$$

i.e. $\mu^*(A) \leqslant \mu^*(F)$.

To prove countable subadditivity suppose that

$$A \subseteq \bigcup_{n=1}^{\infty} A_n,$$

where $A, A_1, A_2, \ldots$ are in $\mathscr{P}(X)$. We must show

$$\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu^*(A_n).$$

Two cases:

(i) if $\sum_{n=1}^{\infty} \mu^*(A_n) = \infty$, the result is trivial;

(ii) otherwise $\sum_{n=1}^{\infty} \mu^*(A_n) < \infty$, so also each $\mu^*(A_n) < \infty$.

Let $\varepsilon > 0$. Because $\mu^*(A_n)$ is finite, we can choose $E_{nk} \in \mathscr{C}$ $(k \in \mathbb{N})$ s.t.

$$A_n \subseteq \bigcup_{k=1}^{\infty} E_{nk}$$

and

$$\left(\sum_{k=1}^{\infty} \mu(E_{nk}) \right) < \mu^*(A_n) + \frac{\varepsilon}{2^n}$$

[by definition of $\mu^*(A_n)$].

Then

$$A \subseteq \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} E_{nk}.$$

N.B.

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{\infty} \mu(E_{nk})\right) &\leqslant \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \frac{\varepsilon}{2^n}\right) \\ &= \left(\sum_{n=1}^{\infty} \mu^*(A_n)\right) + \varepsilon \\ &< \infty.\end{aligned}$$

Thus, the countable sum of all the $\mu(E_{nk})$ is equal to

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \mu(E_{nk}) \leqslant \left(\sum_{n=1}^{\infty} \mu^*(A_n)\right) + \varepsilon.$$

Thus

$$\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu^*(A_n) + \varepsilon$$

since this is true $\forall\ \varepsilon > 0$, we obtain

$$\mu^*(A) \leqslant \sum_{n=1}^{\infty} \mu^*(A_n). \qquad \square$$

Theorem 5.16 (Extension Theorem)

Let R be a ring of subsets of a set X, and let $\mu \colon R \to [0, \infty]$ be a measure on R. Then μ has an extension to a measure $\tilde{\mu}$ defined on a σ-field $\mathscr{F} \supseteq R$.

Proof

Define

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n) \colon A_1, A_2, \ldots \in R,\ E \subseteq \bigcup_{n=1}^{\infty} A_n\right\}.$$

Then the conditions of the Lemma are satisfied, and so μ^* is an outer measure on X. Let $\mathscr{F}$ be the set of μ^* measurable subsets of X. Let $\tilde{\mu}$ be the restriction of μ^* to $\mathscr{F}$. Then $\mathscr{F}$ is a σ-field, and $\tilde{\mu}$ is a measure on $\mathscr{F}$.

It remains to show that $R \subseteq \mathscr{F}$ and that $\tilde{\mu}(A) = \mu(A)\ \forall\ A \in R$.

Let $A \in R$, then

$$A \subseteq A \cup \varnothing \cup \varnothing \cup \dots$$

so $\mu^*(A) \leqslant \mu(A)$.

But, if $A_1, A_2, \dots \in R$ and

$$A \subseteq \bigcup_{n=1}^{\infty} A_n$$

then, since μ is a measure on R, $\mu(A) \leqslant \sum_{n=1}^{\infty} \mu(A_n)$. This holds for all such sequences A_n so $\mu(A) \leqslant \mu^*(A)$. Thus $\mu(A) = \mu^*(A)$.

Finally, we show that A is μ^*-measurable. Let $E \subseteq X$. We must show that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A)$$

as usual we only need to prove that LHS $\geqslant$ RHS. Set

$$S_E = \left\{ \sum_{n=1}^{\infty} \mu(A_n) \colon A_1, A_2, \dots \in R,\ \ E \subseteq \bigcup_{n=1}^{\infty} A_n \right\}.$$

Since $\mu^*(E) = \inf S_E$ it is enough to show that $\mu^*(E \cap A) + \mu^*(E \setminus A)$ is a lower bound for S_E. Let $A_1, A_2, \dots \in R$ with $E \subseteq \bigcup_{n=1}^{\infty} A_n$. Then

$$E \cap A \subseteq \bigcup_{n=1}^{\infty} (A_n \cap A),$$

$$E \setminus A \subseteq \bigcup_{n=1}^{\infty} (A_n \setminus A).$$

So

$$\mu^*(E \cap A) \leqslant \sum_{n=1}^{\infty} \mu(A_n \cap A),$$

$$\mu^*(E \setminus A) \leqslant \sum_{n=1}^{\infty} \mu(A_n \setminus A).$$

Adding gives

$$\begin{aligned} \mu^*(E \cap A) + \mu^*(E \setminus A) &\leqslant \sum_{n=1}^{\infty} \left(\mu(A_n \cap A) + \mu(A_n \setminus A)\right) \\ &= \sum_{n=1}^{\infty} \mu(A_n) \end{aligned}$$

because μ is a measure on R.

This shows $\mu^*(E \cap A) + \mu^*(E \setminus A)$ is a lower bound for S_E. Thus

$$\mu^*(E \cap A) + \mu^*(E \setminus A) \leqslant \mu^*(E).$$

This concludes the proof. □

With our measure $\mu\colon \mathcal{E} \to [0,\infty]$ defined by

$$\mu\left(\dot{\bigcup_{k=1}^{n}} (a_k, b_k]\right) = \sum_{k=1}^{n} (b_k - a_k)$$

we obtain an outer measure μ^* as usual. This particular outer measure is called *Lebesgue* outer measure on $\mathbb{R}$ and we shall denote it by λ^*. The set, $\mathcal{F}$, of subsets of $\mathbb{R}$ which are λ^*-measurable, is known as the collection of Lebesgue measurable subsets of $\mathbb{R}$. Our theorem tells us that $\mathcal{F}$ is a σ-field containing $\mathcal{E}$, and that the restriction of λ^* to $\mathcal{F}$ is a measure. We shall denote this restriction by λ and call it *Lebesgue Measure*. Note that $\mathcal{F}$ contains the σ-field generated by $\mathcal{E}$, which is precisely $\mathcal{B}$, the collection of all Borel sets in $\mathbb{R}$.

With this notation we have

$$P \subseteq \mathcal{E} \subseteq \mathcal{B} \subseteq \mathcal{F} \subseteq \mathcal{P}(\mathbb{R}),$$

and we can measure the size of all Borel sets using λ.

Definition 5.17

Let $(X, \mathcal{F})$ be a measurable space. Then we say that the sets E in $\mathcal{F}$ are *$\mathcal{F}$-measurable* sets (or *measurable* if the σ-field is unambiguous). In particular, if X is a metric space, by default we take $\mathcal{F} = \{E \subseteq X \mid E \text{ is a Borel set}\}$. In this case the measurable sets are the Borel sets in X, so we say such sets are *Borel measurable*. When we have an outer measure μ^* on a set X we already have defined μ^*-measurable. This coincides with the $\mathcal{F}$-measurable sets when $\mathcal{F} = \{E \subseteq X \mid E \text{ is } \mu^*\text{-measurable}\}$. In the particular case of Lebesgue outer measure on $\mathbb{R}$, λ^*, the λ^*-measurable sets are the *Lebesgue measurable sets* (or *Lebesgue sets*) in $\mathbb{R}$.

Every Borel set is Lebesgue measurable. The converse is false but tricky. It turns out that the cardinality of $\mathcal{B}$ is the same as that of $\mathbb{R}$, whereas the cardinality of $\mathcal{F}$ is the same as that of $\mathcal{P}(\mathbb{R})$.

[Recall: Two sets A, B are said to have the same cardinality if there is a bijection $f\colon A \to B$. For every set X, X and $\mathcal{P}(X)$ have different cardinalities.]

Is every subset of $\mathbb{R}$ Lebesgue measurable? The answer is *no*, but most sets you meet *are*.

Proposition 5.18

Let $\mathcal{S}$ be a semi-ring of subsets of a set X, and set $R = R(\mathcal{S})$.

Suppose $\mu\colon R \to [0,\infty]$ is a measure, and we form μ^* as usual by

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n)\colon A_1, A_2 \ldots \in R,\ E \subseteq \bigcup_{n=1}^{\infty} A_n\right\}.$$

Then, in fact, we also have

$$\mu^*(E) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n)\colon A_1, A_2 \ldots \in \mathcal{S},\ E \subseteq \bigcup_{n=1}^{\infty} A_n\right\}.$$

Proof

The sets of extended real numbers

$$S_1 = \left\{ \sum_{n=1}^{\infty} \mu(A_n) \colon A_1, A_2 \ldots \in R,\ \ E \subseteq \bigcup_{n=1}^{\infty} A_n \right\}$$

and

$$S_2 = \left\{ \sum_{n=1}^{\infty} \mu(A_n) \colon A_1, A_2 \ldots \in \mathscr{S},\ \ E \subseteq \bigcup_{n=1}^{\infty} A_n \right\}$$

are in fact the same. The fact that $S_2 \subseteq S_1$ is obvious. Now suppose $x \in S_1$. Then there are $A_1, A_2 \ldots \in R$ with

$$E \subseteq \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad x = \sum_{n=1}^{\infty} \mu(A_n).$$

But we may write

$$A_n = \dot{\bigcup_{k=1}^{m_n}} B_{nk}$$

with B_{nk} in $\mathscr{S}$. Then

$$\mu(A_n) = \sum_{k=1}^{m_n} \mu(B_{nk}) \quad \text{all } n,$$

$$x = \sum_{n=1}^{\infty} \mu(A_n) = \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} \mu(B_{nk}).$$

Since

$$E \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{m_n} B_{nk},$$

a countable union of sets in $\mathscr{S}$, we deduce that $x \in S_2$. This proves our claim.

Corollary 5.19

In the particular case of Lebesgue outer measure λ^*, we find

$$\lambda^*(E) = \inf \left\{ \sum_{n=1}^{\infty} (b_n - a_n) \colon\ E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n] \right\}.$$

Proposition 5.20

(i) For $a, b \in \mathbb{R}$ with $a < b$,

$$\begin{aligned} \lambda([a,b]) &= \lambda([a,b)) = \lambda((a,b)), \\ &= \lambda((a,b]) = b - a. \end{aligned}$$

(ii) For $E \subseteq \mathbb{R}$,

$$\lambda^*(E) = \inf\left\{\sum_{n=1}^{\infty} (b_n - a_n):\ E \subseteq \bigcup_{n=1}^{\infty} (a_n, b_n)\right\}$$
$$= \inf\left\{\sum_{n=1}^{\infty} (b_n - a_n):\ E \subseteq \bigcup_{n=1}^{\infty} [a_n, b_n]\right\}.$$

Proof. See question sheet 5. □

Slightly trickier is to show that λ (Cantor set) = 0.

[Note that the Cantor set is closed, hence is Borel, hence is Lebesgue measurable.]

If $\lambda^*(A) = 0$, then we can show A must be Lebesgue measurable. This is a special case of the following.

Lemma 5.21

Let $\mu^*: \mathscr{P}(X) \to [0, \infty]$ be an outer measure on a set X.

(i) If $A \subseteq X$ and $\mu^*(A) = 0$ then A is μ^*-measurable.

(ii) If B is μ^*-measurable and $A \subseteq B$ and $\mu^*(B) = 0$ then A is μ^*-measurable.

(iii) If $A \subseteq C \subseteq B$ with A, B μ^*-measurable and $\mu^*(B \setminus A) = 0$ then C is μ^*-measurable.

Proof

(i) Given $\mu^*(A) = 0$, let $E \subseteq X$. Then

$$\begin{aligned}\mu^*(E) &\leqslant \mu^*(E \setminus A) + \mu^*(E \cap A),\\ &\leqslant \mu^*(E) + \mu^*(A),\\ &= \mu^*(E).\end{aligned}$$

Thus equality holds and A is μ^*-measurable.

(ii) $A \subseteq B$, and $\mu^*(B) = 0$ then $\mu^*(A) = 0$. So A is μ^*-measurable.

(iii) A, B μ^*-measurable, $A \subseteq C \subseteq B$, $\mu^*(B \setminus A) = 0$.
Then $C \setminus A \subseteq B \setminus A$, so $C \setminus A$ is μ^*-measurable. But $C = (C \setminus A) \cup A$, which is μ^*-measurable.

In particular, given that λ (Cantor set) = 0, we deduce that *every* subset of the Cantor set is Lebesgue measurable. But the Cantor set has the same cardinality as $\mathbb{R}$ (FACT) and so the collection of subsets of the Cantor set has the same cardinality as $\mathscr{P}(\mathbb{R})$. It follows that the collection of Lebesgue sets has the same cardinality as $\mathscr{P}(\mathbb{R})$ [using the Schroder–Bernstein theorem].

Not all of the subsets of the Cantor set are Borel sets (there are too many!). So if you restrict λ to $\mathscr{B}$ you have an *incomplete* measure, i.e. there are sets $A \subseteq B$ with

$$B \in \mathscr{B}, \quad \lambda(B) = 0 \quad \text{but} \quad A \notin \mathscr{B}.$$

Notation

For any $E \subseteq \mathbb{R}$, we write

$$E+x = \{y+x \colon y \in E\} \quad \text{(for any } x \text{ in } \mathbb{R}\text{)}.$$

Proposition 5.22

(i) If $E = (a, b]$ then

$$\lambda(E) = \lambda(E+x) \quad (x \in \mathbb{R}).$$

(ii) If $E \subseteq \mathbb{R}$ then

$$\lambda^*(E+x) = \lambda^*(E) \quad (x \in \mathbb{R}).$$

(iii) If $A \subseteq \mathbb{R}$ is Lebesgue measurable, then so is $A+x$ $(x \in \mathbb{R})$, and

$$\lambda(A+x) = \lambda(A).$$

(iv) If $A \subseteq \mathbb{R}$ is Lebesgue measurable, then so is $-A$, and

$$\lambda(-A) = \lambda(A).$$

Proof

(i) $E+x = (a+x, b+x]$ and $(b+x)-(a+x) = b-a$.

(ii) This is an easy exercise, based on (i) and the definition of λ^*.

(iii)

$$\lambda^*(E-x) = \lambda^*(E) = \lambda^*(E \cap A) + \lambda^*(E \setminus A)$$

$$E \cap (A+x) = ((E-x) \cap A) + x,$$
$$E \setminus (A+x) = ((E-x) \setminus A) + x.$$

$$\lambda^*(E) = \lambda^*(E-x) = \lambda^*((E-x) \cap A) + \lambda^*((E-x) \setminus A) \qquad \text{because } A \text{ is measurable}$$
$$= \lambda^*(E \cap (A+x)) + \lambda^*(E \setminus A+x).$$

This is true for all $E \subseteq X$, and so $A+x$ is Lebesgue measurable.

By (ii),

$$\lambda(A+x) = \lambda(A).$$

(iv) This is similar to (i)–(iii), using $-(a, b) = (-b, -a)$, and the definition of λ^* in terms of open intervals. □

So Lebesgue measure is *translation invariant*.

A non-measurable set

To find a non-measurable set we will need the axiom of choice.

Example 5.23

Working in $[0,1]$ we define an equivalence relation $\sim$ by $x \sim y$ if $x - y \in \mathbb{Q}$. This equivalence relation splits $[0,1]$ up into equivalence classes. Note, for $x \in [0,1]$, the equivalence class of x is $(\mathbb{Q} + x) \cap [0,1]$.

We now form a set E by choosing *one* element from each equivalence class. [This uses the axiom of choice.]

We show that E is not Lebesgue measurable.

Set $S = \mathbb{Q} \cap [-1,1]$, a countable set. Then

$$[0,1] \subseteq \bigcup_{q \in S} (E+q) \subseteq [-1,2]$$

because $E \subseteq [0,1]$, and, $\forall\, y \in [0,1]$, $\exists\, x \in E$ with $(y-x) \in \mathbb{Q}$ since x, y are in $[0,1]$, we have then $(y-x) \in S$, and $y \in E + (y-x)$.

The collection of sets $\{E+q : q \in S\}$ is countable, and pairwise disjoint.

(Reason: if $p \neq q \in S$, then suppose we had

$$x \in (E+p) \cap (E+q).$$

Then we would have $(x-p) \in E$ and $(x-q) \in E$. This is impossible because E has only *one* member of each equivalence class.)

By assumption, E is measurable, and so $E+q$ is measurable $\forall\, q \in S$.

We have

$$1 = \lambda([0,1]) \leqslant \lambda\left(\bigcup_{q \in S} (E+q)\right) = \sum_{q \in S} \lambda(E+q) = \sum_{q \in S} \lambda(E).$$

Also

$$\lambda\left(\bigcup_{q \in S} (E+q)\right) \leqslant \lambda([-1,2]) = 3$$

so

$$\sum_{q \in S} \lambda(E) \leqslant 3.$$

This is impossible: $\sum_{q \in S} \lambda(E)$ must be either 0 or ∞.

Regularity of Lebesgue Measure

Recall: (see question sheet again)

$$\lambda^*(E) = \inf\left\{\sum_{n=1}^{\infty}(b_n - a_n):\ E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)\right\}.$$

Lemma 5.24

For any $E \subseteq \mathbb{R}$, $\lambda^*(E) = \inf\{\lambda(U)\colon U \text{ open}, U \supseteq E\}$.

Proof

Certainly $\lambda^*(E) \leqslant \inf\{\lambda(U)\colon U \text{ open}, U \supseteq E\}$.

Suppose $x > \lambda^*(E)$. Then $\exists\ (a_1, b_1), (a_2, b_2), \ldots$ such that

$$E \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n) \quad \text{and} \quad \sum_{n=1}^{\infty}(b_n - a_n) < x.$$

Set $V = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Then $\lambda(V) \leqslant \sum_{n=1}^{\infty}(b_n - a_n) < x$. Also $E \subseteq V$, and so we have

$$\inf\{\lambda(U)\colon U \text{ open},\ U \supseteq E\} < x.$$

This is true $\forall\ x > \lambda^*(E)$ and so the result follows. □

N.B. We do not claim to be able to obtain

$$\lambda^*(U \setminus E) < \varepsilon.$$

Theorem 5.25

Let E be a Lebesgue measurable subset of $\mathbb{R}$. Then

(i) $\forall\ \varepsilon > 0$ there is an open set $U \subseteq \mathbb{R}$ with

$$\lambda(U \setminus E) < \varepsilon \quad \text{and} \quad E \subseteq U$$

(ii) $\forall\ \varepsilon > 0\ \exists$ a closed set $F \subseteq E$ with

$$\lambda(E \setminus F) < \varepsilon.$$

Proof

(i) Set $E_n = E \cap [-n, n]$ for $n \in \mathbb{N}$. Then E_n is measurable, and $\lambda(E_n) < \infty$. Choose open sets V_n with $E_n \subseteq V_n$ satisfying $\lambda(V_n) < \lambda^*(E_n) + \dfrac{\varepsilon}{2^n}$. We can do this because $\lambda^*(E_n) < \infty$. Then set $U = \bigcup_{n=1}^{\infty} V_n$. Certainly U is open, and

$$U \setminus E = \bigcup_{n=1}^{\infty} (V_n \setminus E) \subseteq \bigcup_{n=1}^{\infty} (V_n \setminus E_n).$$

Thus

$$\lambda(U \setminus E) \leqslant \sum_{n=1}^{\infty} \lambda(V_n \setminus E_n) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon$$

(because the sets E_n are measurable) proving (i).

(ii) To prove (ii), we use (i), and set $A = E^c$. By (i) $\exists$ an open set $U \supseteq A$ with $\lambda(U \setminus A) < \varepsilon$.

Set $F = \mathbb{R} \setminus U$. Then $F \subseteq \mathbb{R} \setminus A = E$ and

$$(E \setminus F) = (\mathbb{R} \setminus F) \setminus (\mathbb{R} \setminus E) = U \setminus A$$

so

$$\lambda(E \setminus F) = \lambda(U \setminus A) < \varepsilon. \qquad \square$$

The fact that you can approximate Lebesgue measurable sets from the inside by closed sets, and from the outside by open sets, is described by saying that Lebesgue measure λ is *regular*.

If $(X, \mathscr{F})$ is a measurable space and $E \in \mathscr{F}$ then

$$\{F \cap E : F \in \mathscr{F}\} = \{F \in \mathscr{F} : F \subseteq E\}$$

is a σ-field on E, denoted by $\mathscr{F}|_E$ (non-standard notation) or $\mathscr{F}_E$.

If $\mu : \mathscr{F} \to [0, \infty]$ is a measure, then $\mu|_{\mathscr{F}_E}$ is a measure on $\mathscr{F}_E$. We will sometimes denote this measure by μ_E.

For any Lebesgue measurable set $E \subseteq \mathbb{R}$, we have a σ-field consisting of all Lebesgue measurable sets contained in E, and we can restrict Lebesgue measure to this. In particular we can work on any interval $[a, b]$.

Exercise

Regarding $[a, b]$ as a metric space, show that the Borel subsets of $[a, b]$ are precisely the sets in $\mathscr{B}_{[a,b]}$ (where $\mathscr{B}_{[a,b]} = \{E \cap [a, b] : E \in \mathscr{B}\} = \{E \in \mathscr{B} : E \subseteq [a, b]\}$). So 'Borel subsets of $[a, b]$' is unambiguous.

In particular, restricting attention to $[0, 1]$, Lebesgue measure gives a probability measure on the set of Borel subsets of $[0, 1]$.