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HG1M12: Engineering Mathematics 2

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Outline

- An ordinary differential equation (ODE) is an equation involving an unknown function, $y(x)$ say, and its derivatives; for example,

$$\frac{dy}{dx} = y, \qquad \text{or} \qquad \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 6y = 0.$$

- **AIM**: to learn how to solve ODEs to determine the unknown function $y(x)$.

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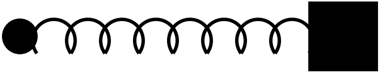
- **AIM**: to learn how to solve ODEs to determine the unknown function $y(x)$.
- Notation: y is called the 'dependent' variable, and x is the 'independent' variable.
  Other notations are possible, such as an ODE for x as a function of t, e.g. $3\dfrac{d^2x}{dt^2} + 5\dfrac{dx}{dt} + 7x = 0.$
- ODEs arise quite naturally in a wide variety of practical engineering situations. Some examples of this now follow...

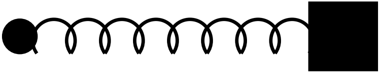
3.1.1 Motion of a mass on a spring in a resistive medium



$$x = 0 \qquad x = X(t)$$

Resistive force: $-k_1 \times (\text{speed}) = -k_1 \dfrac{dX}{dt}$,
Spring restoring force: $-k_2 \times (\text{extension}) = -k_2(X - l)$,
where k_1 and k_2 are constants and l is the natural length of the spring.

3.1.1 Motion of a mass on a spring in a resistive medium



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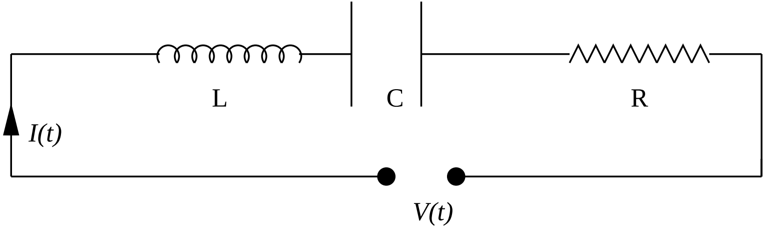
Spring restoring force: $-k_2 \times$ (extension) $= -k_2(X - l)$,
where k_1 and k_2 are constants and l is the natural length of the spring.
Newton's Second Law ($F = ma = m\ddot{x}$) $\Rightarrow$

$$-k_1 \frac{dX}{dt} - k_2(X - l) = m \frac{d^2X}{dt^2}$$

$$\text{i.e.} \quad m\frac{d^2X}{dt^2} + k_1 \frac{dX}{dt} + k_2 X = k_2 l$$

3.1.2 The LCR circuit



Kirchoff's Law: $$L\frac{dI}{dt} + IR + \frac{1}{C}\int I(t)dt = V(t)$$

Differentiating gives $L\dfrac{d^2I}{dt^2} + R\dfrac{dI}{dt} + \dfrac{1}{C}I = \dfrac{dV}{dt}$

— an ODE for $I(t)$.

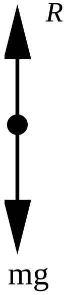
3.1.3 Newton's law of cooling

The rate of decrease of temperature of a body is proportional to the temperature difference between the body and the surrounding air

$$\text{that is,} \quad \frac{dT}{dt} \propto T - T_0 \text{or, equivalently} \quad \frac{dT}{dt} = -k(T - T_0)$$

for some constant k.

3.1.4 Free fall under gravity with air resistance



A mass m falls a distance $y(t)$,

and has downward speed $V(t) = \dfrac{dy}{dt}$

Resistive force, $R \propto V^2 \Rightarrow R = kV^2, (k > 0)$

$$F = ma \Rightarrow mg - R = m\frac{dV}{dt}$$

$$\Rightarrow \quad \boxed{m\frac{dV}{dt} = mg - kV^2} \quad \text{for} \quad V = V(t).$$

We can rewrite this in other ways: • for y as a function of t, using $y(t)$

$$V = \frac{dy}{dt} \quad \Rightarrow \quad \boxed{m\frac{d^2y}{dt^2} = mg - k\left(\frac{dy}{dt}\right)^2};$$

• for V as a function of y, that is $V(y)$,

$$\text{using} \quad \frac{d}{dt} = \frac{dy}{dt}\frac{d}{dy} = V\frac{d}{dy} \quad \text{implies} \quad \boxed{mV\frac{dV}{dy} = mg - kV^2}$$

3.2 Order of an ODE

The order of an ODE is the highest derivative occurring in the equation. For example,

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} - 10y = 0 \quad \text{is } \textbf{second}\text{-order;}$$

$$3\frac{dy}{dx} + 4y^3x = 0 \quad \text{is } \textbf{first}\text{-order.}$$

We only consider first and second-order ODEs in this module, though some of the ideas carry over to higher-order examples.

3.3 Linearity

- An ODE is '**linear**' if (a) the only y-dependent terms are y itself and derivatives of y and (b) these terms do not appear multiplied together.
- ODEs containing products of y-dependent terms, or functions of y, are said to be '**nonlinear**'.

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- $\dfrac{d^2y}{dx^2} + 10\dfrac{dy}{dx} - 6y = 0$ is second-order and **linear**;

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- $\dfrac{d^2y}{dx^2} + 10\dfrac{dy}{dx} - 6y = 0$ is second-order and **linear**;
- $y\dfrac{dy}{dx} + x^3 = 0$ is first-order and **nonlinear** (due to the term $y\frac{dy}{dx}$),

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  (due to the $(\frac{dy}{dx})^3$ term), whilst
- $\dfrac{d^2y}{dx^2} + 10\dfrac{dy}{dx} - 6y = 0$ is second-order and **linear**;
- $y\dfrac{dy}{dx} + x^3 = 0$ is first-order and **nonlinear**
  (due to the term $y\frac{dy}{dx}$), whilst
- $\dfrac{dy}{dx} + 4x^3y = e^{4x}$ is first-order and **linear**.

3.4 First-order ODEs

Notice that the coefficients in the ODE might depend upon the independent variable (x in this case).

- First-order ODEs are usually written in the canonical form

$$\boxed{\frac{dy}{dx} = f(x,y)}$$

where f is a given function. Sometimes they are written in the equivalent form

$$y'(x) = f(x,y).$$

- Not all ODEs can be solved explicitly, we now cover some techniques which enable certain classes to be solved.

3.5 Separable first-order ODEs

In this case, the function f 'separates' into the product of a function of x and a function of y. Then we can write $f(x, y) = g(x)h(y)$

so that the equation becomes $\dfrac{dy}{dx} = g(x)h(y)$.

We can then rearrange this into the form $\dfrac{1}{h(y)}\dfrac{dy}{dx} = g(x)$.

Hence, the solution is obtained by $\displaystyle\int \frac{1}{h(y)}dy = \int g(x)dx + c,$

remembering to include the arbitrary constant of integration c.

3.5.1 Example

Find the general solution of $\dfrac{dy}{dx} = ky$.

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Find the general solution of $\dfrac{dy}{dx} = ky$.

Solution

$$\frac{dy}{dx} = ky \quad \Rightarrow \quad \int \frac{1}{y} dy = \int k dx + c \quad \Rightarrow \quad \ln y = kx + c.$$

Hence,

$$y = e^{kx+c} = e^c e^{kx} = Ae^{kx},$$

where $A = e^c$ is an arbitrary constant.

3.5.2 Example

Find the general solution of $\dfrac{dy}{dx} = y^{1/2}x.$

Find the general solution of $\dfrac{dy}{dx} = y^{1/2}x$.

Solution.

$$\frac{dy}{dx} = y^{\frac{1}{2}}x \quad \Rightarrow \quad \int \frac{dy}{y^{\frac{1}{2}}} = \int xdx + c.$$

Hence,

$$2y^{1/2} = \frac{x^2}{2} + c \quad \text{or} \quad y^{1/2} = \frac{1}{4}\left(x^2 + A\right),$$

where $A = 2c$ is an arbitrary constant. Finally,

$$y = \frac{1}{16}\left(x^2 + A\right)^2.$$

Find the general solution of $$(1 - x^2)\frac{dy}{dx} = -2xe^{-y}$$

3.5.3 Example

Find the general solution of $\quad (1-x^2)\dfrac{dy}{dx} = -2xe^{-y}$

Solution This is separable - divide by $(1-x^2)$ and multiply by e^y. In this case, when the variables have been separated we get

$$\int e^y\,dy = \int \frac{-2x}{(1-x^2)}dx + c.$$

$$\text{Hence,} \qquad e^y = \ln|1-x^2| + c$$

or, if you prefer,

$$y = \ln\left(\ln|1-x^2| + c\right).$$

3.5.4 Implicit solutions

It is not always possible to get an explicit expression for y as a function of x. For example:

$$\frac{dy}{dx} = \frac{\cos x}{\sin y}$$

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separates to give

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which has the ‘solution’

$$-\cos y = \sin x + c.$$

We consider this as the solution, and it represents a family of curves with each member of the family corresponding to a particular value of the constant of integration, c.

3.6 First order linear equations

Recalling our definition of a **linear** differential equation, we see that some of the separable cases that we have studied are nonlinear. For

$$\frac{dy}{dx} = f(x, y)$$

to be linear we must have

$$f(x, y) = -p(x)y + q(x)$$

for given functions p and q. Then the equation becomes

$$\frac{dy}{dx} + p(x)y = q(x).$$

No 'y'-terms are multiplied together and there are no functions of y.

3.6.1 Homogeneous first-order linear ODEs

If $q(x) = 0$, the equation is said to be **homogeneous.**
If $q(x) = 0$ the equation is also separable. Then

$$\frac{dy}{dx} + p(x)y = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -p(x)y.$$

Hence,

$$\int \frac{1}{y} dy = -\int p(x)dx + k,$$

where k is a constant of integration, so that

$$\ln y = -\int p(x)dx + k.$$

Taking exponentials, $y = ce^{-\int p(x)dx}$
where $c = e^k$ is an arbitrary constant.

3.6.2 An observation

$$
\begin{aligned}
\frac{d}{dx}\left(y(x)\, e^{\int p(x)dx}\right) &= e^{\int p(x)dx}\frac{dy}{dx} + y\frac{d}{dx}e^{\int p(x)dx} \\
&= e^{\int p(x)dx}\frac{dy}{dx} + ye^{\int p(x)dx}\left(\frac{d}{dx}\int p(x)dx\right) \\
&= e^{\int p(x)dx}\frac{dy}{dx} + e^{\int p(x)dx}p(x)y \\
&= e^{\int p(x)dx}\left(\frac{dy}{dx} + p(x)y\right).
\end{aligned}
$$

Hence:

$$
e^{\int p(x)dx}\left(\frac{dy}{dx} + p(x)y\right) = \frac{d}{dx}\left(ye^{\int p(x)dx}\right).
$$

To solve the ODE $\quad \dfrac{dy}{dx} + p(x)y = q(x) \quad$,

we multiply both sides by the **integrating factor (IF)** $e^{\int p(x)dx}$:

$$e^{\int p(x)dx}\left(\frac{dy}{dx} + p(x)y\right) = e^{\int p(x)dx}q(x).$$

From the previous result, we have

$$\frac{d}{dx}\left(ye^{\int p(x)dx}\right) = q(x)e^{\int p(x)dx}.$$

3.6.3 The inhomogeneous case

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From the previous result, we have

$$\frac{d}{dx}\left(ye^{\int p(x)dx}\right) = q(x)e^{\int p(x)dx}.$$

Now integrate both sides with respect to x:

$$ye^{\int p(x)dx} = \int q(x)e^{\int p(x)dx}dx + c.$$

$\Rightarrow$ Final result: $\boxed{y = e^{-\int p(x)dx} \int q(x)e^{\int p(x)dx}dx + ce^{-\int p(x)dx}}.$

NB. If $q(x) = 0$, then we get the same result as before.

3.6.4 Example

Find the general solution of $\dfrac{dy}{dx} - 2y = e^x$.

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Find the general solution of $\dfrac{dy}{dx} - 2y = e^x$.

Solution. We identify $\quad p(x) = -2, \quad q(x) = e^x$

The integrating factor is $\quad e^{\int p(x)dx} = e^{-\int 2dx} = e^{-2x}$

$$\text{Hence,} \quad e^{-2x}\left(\frac{dy}{dx} - 2y\right) = e^{-2x}e^x = e^{-x}.$$

Then

$$\frac{d}{dx}\left(ye^{-2x}\right) = e^{-x} \quad \Rightarrow \quad ye^{-2x} = \int e^{-x}dx + c$$

$$\Rightarrow \quad ye^{-2x} = -e^{-x} + c \quad \Rightarrow \quad y = -e^{-x}e^{2x} + ce^{2x}.$$

$$\Rightarrow \quad y = -e^x + ce^{2x} \quad \text{for arbitrary constant } c.$$

3.6.5 Example

Find the general solution of $$x\frac{dy}{dx} + 2y = 3x^3$$

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Find the general solution of $\quad x\dfrac{dy}{dx} + 2y = 3x^3$

Solution: divide by the coefficient of $\dfrac{dy}{dx}$ to obtain $\dfrac{dy}{dx} + \dfrac{2}{x}y = 3x^2$ (1)

Identify $p(x) = 2/x$ and hence find the **integrating factor**:

$$e^{\int p(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = e^{\left(\ln x^2\right)} = x^2.$$

Multiply (1) by **IF** = x^2:

$$x^2\frac{dy}{dx} + 2xy = 3x^4 \quad \Rightarrow \quad \frac{d}{dx}\left(x^2 y\right) = 3x^4$$

$$\Rightarrow \quad x^2 y = \frac{3}{5}x^5 + c \quad \Rightarrow \quad y = \frac{3}{5}x^3 + \frac{c}{x^2}.$$

3.6.6 Boundary and initial conditions

- We have seen that the solution for a **first**-order ODE involves **one** unknown constant. This is called the **general solution**.
- Generally, the number of unknown constants in the general solution equals the order of the ODE.
- To determine a specific solution, we must find these constants.
- We therefore need as many conditions on the solution as there are unknown constants.
- These conditions usually involve the value of the unknown function and/or some of its derivatives at certain points.
- For a first-order equation we only need one such condition, and this is called a **boundary condition**.
- If the independent variable in the ODE is time t, then we normally specify the condition at time $t = 0$. Then the boundary condition is called an **initial condition**.

3.6.7 Example

Find the general solution of $\dfrac{dy}{dx} + \dfrac{x}{1+x^2}y = x$
and the solution subject to the boundary condition $y(1) = 0$.

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Find the general solution of $\dfrac{dy}{dx} + \dfrac{x}{1+x^2}y = x$

and the solution subject to the boundary condition $y(1) = 0$.

Solution. This is a linear ODE with

$$p(x) = \frac{x}{1+x^2}, \qquad q(x) = x.$$

Hence we find the integrating factor

$$\int p(x)dx = \int \frac{x}{1+x^2}dx = \frac{1}{2}\ln\left(1+x^2\right) = \ln\left(1+x^2\right)^{1/2}.$$

$$\Rightarrow \quad e^{\int p(x)dx} = e^{\ln\left(1+x^2\right)^{1/2}} = \left(1+x^2\right)^{1/2}.$$

The effect of multiplying the ODE by the integrating factor is to put the equation into the form

$$\frac{d}{dx}\left(y\left(1+x^2\right)^{1/2}\right) = x\left(1+x^2\right)^{1/2}$$

3.6.7 Example (ctd)

$$\Rightarrow \quad y\left(1+x^2\right)^{1/2} = \int x\left(1+x^2\right)^{1/2} dx + c = \frac{1}{3}\left(1+x^2\right)^{3/2} + c,$$

$$\Rightarrow \quad y = \frac{1}{3}\left(1+x^2\right) + c\left(1+x^2\right)^{-1/2}.$$

This is the general solution to the ODE.

3.6.7 Example (ctd)

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$$\Rightarrow \quad y = \frac{1}{3}\left(1+x^2\right) + c\left(1+x^2\right)^{-1/2}.$$

This is the general solution to the ODE.
(ii) Suppose that we insist that $y = 0$ when $x = 1$. i.e. $y(1) = 0$. Then

$$\frac{1}{3}(1+1) + c(1+1)^{-\frac{1}{2}} = 0$$

which can be solved to give the value of c in this case as $\quad c = -\dfrac{2\sqrt{2}}{3}$.

The solution for y is then

$$y = \frac{1}{3}\left(1+x^2\right) - \frac{2\sqrt{2}}{3}\left(1+x^2\right)^{-1/2}.$$

3.6.8 Example

Suppose that a variable current $I(t)$, where t is time in seconds, flows through a coil with inductance L and a resistor of resistance R with an applied voltage V, where we assume that L, R and V are constants. Suppose also that the current is zero at time $t = 0$. Find $I(t)$.

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Solution The current satisfies the ODE $\quad L\dfrac{dI}{dt} + RI = V$

subject to the initial condition $\quad I(0) = 0$.

We can re-write the ODE in the form $\quad \dfrac{dI}{dt} + \dfrac{R}{L}I = \dfrac{V}{L},$

which is linear, and first-order.

Since $p = R/L$, the integrating factor is $\exp\left(\int \frac{R}{L}dt\right) = e^{Rt/L}$.

Hence, $e^{Rt/L}\left(\frac{dI}{dt}+\frac{R}{L}I\right)=\frac{V}{L}e^{Rt/L} \;=\; \frac{d}{dt}\left(e^{Rt/L}I\right)=\frac{V}{L}e^{Rt/L}$

$$\Rightarrow \quad e^{Rt/L}I=\frac{V}{L}\frac{L}{R}e^{Rt/L}+c,$$

where c is a constant of integration. Hence, the general solution for I is

$$I(t)=\frac{V}{R}+ce^{-Rt/L}.$$

3.6.8 Example (ctd)

$$\text{Hence,} \quad e^{Rt/L}\left(\frac{dI}{dt}+\frac{R}{L}I\right)=\frac{V}{L}e^{Rt/L} \;=\; \frac{d}{dt}\left(e^{Rt/L}I\right)=\frac{V}{L}e^{Rt/L}$$

$$\Rightarrow \quad e^{Rt/L}I=\frac{V}{L}\frac{L}{R}e^{Rt/L}+c,$$

where c is a constant of integration. Hence, the general solution for I is

$$I(t)=\frac{V}{R}+ce^{-Rt/L}.$$

But $I(0)=0$, from the initial condition. So $\quad 0=\frac{V}{R}+c \;\Rightarrow\; c=-\frac{V}{R}.$

Therefore the solution for $I(t)$ is $\quad I(t)=\frac{V}{R}\left(1-e^{-Rt/L}\right).$

3.6.9 Example

The rate of increase of the concentration of a chemical is proportional to the concentration at that time.
Experiment shows that the concentration doubles in four hours.
If the initial concentration is c_0, what is the concentration after five hours?

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Solution: Let $c(t)$ denote the concentration at time t. Then $\dfrac{dc}{dt} = kc$, for some constant k. This is a separable ODE and so we obtain

$$\int \frac{1}{c} dc = \int k dt + A \quad \Rightarrow \quad \ln c(t) = kt + A \quad \Rightarrow \quad c(t) = Be^{kt}$$

where A and $B = e^A$ are constants.

3.6.9 Example (ctd)

Using $c = Be^{kt}$ and the initial condition

$$c = c_0 \quad at \quad t = 0 \Rightarrow c_0 = Be^0 = B,$$

gives

$$c(t) = c_0 e^{kt}.$$

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gives

$$c(t) = c_0 e^{kt}.$$

At $t = 4$ (in units of hours), $c = 2c_0$. So

$$2c_0 = c_0 e^{4k}$$

or

$$k = \frac{1}{4}\ln 2 \quad \Rightarrow c(t) = c_0 e^{(t/4)\ln 2}.$$

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or

$$k = \frac{1}{4} \ln 2 \quad \Rightarrow c(t) = c_0 e^{(t/4) \ln 2}.$$

If $t = 5$,

$$c(5) = c_0 e^{(5/4) \ln 2} = c_0 e^{\ln 2^{5/4}} = 2^{5/4} c_0.$$

3.7 Exact equations

- We can always express the equation

$$\frac{dy}{dx} = f(x, y)$$

in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0,$$

by setting $f(x, y) = -\frac{M(x,y)}{N(x,y)}$, if necessary.

- Sometimes the left-hand side in the above equation is an *exact derivative*, in which case the ODE is called **exact** and its solution is easily found. This happens if there exists a function $F(x, y)$ such that

$$\frac{dF(x, y)}{dx} = M(x, y) + N(x, y)\frac{dy}{dx}.$$

3.7.1 Example

$$y + x\frac{dy}{dx} = 0 \quad \Leftrightarrow \quad \frac{d}{dx}(xy) = 0,$$

by the product rule (in this case, $F = xy$). Hence, the solution is
$xy = c$,
where c is an arbitrary constant; thus $\quad y = c/x$.

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$xy = c$,
where c is an arbitrary constant; thus $\quad y = c/x$.

Example: $$-\frac{x^2}{y^2}\frac{dy}{dx} + \frac{2x}{y} = 0 \quad \Rightarrow \quad \frac{d}{dx}\left(\frac{x^2}{y}\right) = 0. \quad \Rightarrow \quad \frac{x^2}{y} = c,$$

where c is an arbitrary constant, hence $\quad y = \dfrac{x^2}{c} = Ax^2, \;\; A = \dfrac{1}{c}.$

3.7.2 Conditions for exact derivatives

$$\text{Suppose} \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\text{is equivalent to} \quad \frac{d}{dx}\left(F(x,y)\right) = 0. \qquad (*)$$

Then since

$$\frac{d}{dx}\left(F(x,y(x))\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

we must have that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N.$$

$$\text{Suppose} \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\text{is equivalent to} \quad \frac{d}{dx}\left(F(x,y)\right) = 0. \qquad (*)$$

Then since

$$\frac{d}{dx}\left(F(x,y(x))\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

we must have that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N.$$

$$\text{Since} \quad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}, \quad \text{we must also have} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

3.7.2 Conditions for exact derivatives

$$\text{Suppose} \quad M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$\text{is equivalent to} \quad \frac{d}{dx}\left(F(x,y)\right) = 0. \qquad (*)$$

Then since

$$\frac{d}{dx}\left(F(x,y(x))\right) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx}$$

we must have that

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N.$$

$$\text{Since} \quad \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}, \quad \text{we must also have} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Integrating $(*)$, the solution is $\boxed{F(x,y) = c}$, where c is a constant.

3.7.3 Repeated Example

We revisit the problem: $y + x\dfrac{dy}{dx} = 0$

$$M(x,y) = y, \;\; N(x,y) = x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}.$$

$$M = F_x, N = F_y$$

We revisit the problem: $y + x\dfrac{dy}{dx} = 0$

$$M(x,y) = y, \quad N(x,y) = x \quad \Rightarrow \quad \frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}.$$

$$M = F_x,\ N = F_y \text{ also implies } \quad \frac{\partial F}{\partial x} = y \quad \text{and} \quad \frac{\partial F}{\partial y} = x.$$

$$\frac{\partial F}{\partial x} = y \quad \Rightarrow \quad F = xy + g(y).$$

$$\frac{\partial F}{\partial y} = x \quad \Rightarrow \quad \frac{\partial}{\partial y}(xy + g(y)) = x$$

$$\Rightarrow \quad g'(y) = 0 \quad \text{or} \quad g(y) = \text{const.}$$

$$\text{So} \quad F(x,y) = xy + \text{constant}$$

and the general solution is $\quad xy = \text{constant}, \quad$ as before.

3.7.4 Repeated Example

Let us revisit the problem $-\dfrac{x^2}{y^2}\dfrac{dy}{dx}+\dfrac{2x}{y}=0$

3.7.4 Repeated Example

Let us revisit the problem $-\dfrac{x^2}{y^2}\dfrac{dy}{dx} + \dfrac{2x}{y} = 0$

$$\text{Hence} \quad M(x,y) = \frac{2x}{y}, \quad N(x,y) = -\frac{x^2}{y^2}.$$

$$\text{Check solvability:} \quad \frac{\partial M}{\partial y} = -\frac{2x}{y^2} = \frac{\partial N}{\partial x}.$$

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$$\text{Check solvability:} \quad \frac{\partial M}{\partial y} = -\frac{2x}{y^2} = \frac{\partial N}{\partial x}.$$

$$\text{Solve:} \quad \frac{\partial F}{\partial x} = \frac{2x}{y} \quad \Rightarrow \quad F = \frac{x^2}{y} + g(y). \qquad (*)$$

$$\frac{\partial F}{\partial y} = -\frac{x^2}{y^2} \quad \Rightarrow \quad \frac{\partial}{\partial y}\left(\frac{x^2}{y} + g(y)\right) = -\frac{x^2}{y^2}.$$

$$\text{Hence,} \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = \text{constant}.$$

3.7.4 Repeated Example

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$$\frac{\partial F}{\partial y} = -\frac{x^2}{y^2} \quad \Rightarrow \quad \frac{\partial}{\partial y}\left(\frac{x^2}{y} + g(y)\right) = -\frac{x^2}{y^2}.$$

$$\text{Hence,} \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = \text{constant}.$$

Therefore, from $(*)$, $F(x,y) = x^2/y +$ constant and so the general solution $F(x,y) =$ constant implies $x^2/y =$ constant.

3.7.5 New example

Check the solvability and, if possible, solve $2xy + \left(x^2 - 1\right)\dfrac{dy}{dx} = 0$

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Solution: $M(x,y) = 2xy, \quad N(x,y) = x^2 - 1 \quad \Rightarrow \quad \dfrac{\partial M}{\partial y} = 2x = \dfrac{\partial N}{\partial x}.$

3.7.5 New example

Check the solvability and, if possible, solve $2xy + \left(x^2 - 1\right)\dfrac{dy}{dx} = 0$

Solution: $M(x,y) = 2xy, \quad N(x,y) = x^2 - 1 \quad \Rightarrow \quad \dfrac{\partial M}{\partial y} = 2x = \dfrac{\partial N}{\partial x}.$

We know that the equation must be equivalent to $\dfrac{d}{dx}(F(x, y(x)) = 0$

$$\text{where} \quad \frac{\partial F}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial F}{\partial y} = x^2 - 1.$$

$$\frac{\partial F}{\partial x} = 2xy \quad \Rightarrow \quad F = x^2 y + g(y).$$

$$\text{Therefore} \quad \frac{\partial F}{\partial y} = x^2 + g'(y) \quad = x^2 - 1$$

$$\Rightarrow \quad g'(y) = -1, \quad \text{implying} \quad g(y) = -y + \text{const.}$$

So $F(x,y) = \text{constant} \Rightarrow x^2 y - y = c$.

This can be re-expressed in the form $\quad y = \dfrac{c}{x^2 - 1}.$

3.8 Second-order ODEs

We restrict attention to **second-order** ODEs of the general form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x),$$

where a, b and c are given constants.

- These are called *constant coefficient, linear second-order ODEs*.
- If $f(x) = 0$, the right-hand side is zero and the ODE is referred to as **homogeneous**; we consider these first.
- Other notation is possible; for example, the ODE might be in terms of $x(t)$ and its derivatives with respect to t.

3.8.1 Simpler case

The case $a = 0$.

$$b\frac{dy}{dx} + cy = 0$$

and we know how to solve this (by separating the variables). The solution is

$$y = Ae^{-cx/b}.$$

3.8.1 Simpler case

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and we know how to solve this (by separating the variables). The solution is

$$y = Ae^{-cx/b}.$$

Alternatively, try $y = Ae^{mx}$ as a solution.

$$b\frac{dy}{dx} + cy = 0 \quad \Rightarrow \quad bmAe^{mx} + cAe^{mx} \quad \Rightarrow \quad Ae^{mx}(bm + c) = 0.$$

Hence,

$$m = -\frac{c}{b}$$

and we have obtained the same answer as before.

3.9 Homogeneous 2$^{\text{nd}}$-order ODEs-auxiliary eq

Suppose we adopt the same approach for the second-order equation:

$$y = Ae^{mx} \quad \Rightarrow \quad \frac{dy}{dx} = mAe^{mx}, \quad \frac{d^2y}{dx^2} = m^2Ae^{mx}.$$

Substituting into the ODE, we find

$$am^2Ae^{mx} + bmAe^{mx} + cAe^{mx} = 0$$

$$\Rightarrow \quad Ae^{mx}(am^2 + bm + c) = 0.$$

$$Ae^{mx} \neq 0 \quad \Rightarrow \quad am^2 + bm + c = 0.$$

So m satisfies a quadratic equation called the **auxiliary equation.**

3.9 Auxiliary equation (ctd)

If the two roots are m_1 and m_2, then the general solution for y is

$$\boxed{y = Ae^{m_1 x} + Be^{m_2 x}},$$

where A and B are arbitrary constants.

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Example: Solve $\quad \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$

$$y = Ae^{mx} \quad \Rightarrow \quad Ae^{mx}(m^2 - m - 2) = 0$$

$$\Rightarrow \quad m^2 - m - 2 = 0 \quad \Rightarrow \quad (m+1)(m-2) = 0$$

$$\Rightarrow \quad m_1 = -1, \quad m_2 = 2.$$

The general solution for y is $\quad y = Ae^{-x} + Be^{2x}$
for arbitrary constants A and B.

3.9.1 Example

Solve $$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

3.9.1 Example

Solve $\quad \dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 0$

Solution: $\quad y = Ae^{mx} \quad \Rightarrow \quad Ae^{mx}(m^2 - 4m + 3) = 0$

$$\Rightarrow \quad m^2 - 4m + 3 = 0 \quad \Rightarrow \quad (m-3)(m-1) = 0$$

$$\Rightarrow \quad m_1 = 3, \quad m_2 = 1.$$

The general solution for y is $\quad y = Ae^{3x} + Be^{x}$
for arbitrary constants A and B.

3.9.2 Three types of solution of the auxiliary equation

Since m_1 and m_2 are the roots of a quadratic equation, three different cases are possible:

1. m_1 and m_2 are real and distinct (as in the previous two examples)
2. m_1 and m_2 are real and repeated (i.e. $m_1 = m_2$)
3. m_1 and m_2 are complex conjugates

We now consider the second and third cases, the first having already been dealt with.

3.9.3 Real and repeated roots

2. If $m_1 = m_2$, the general solution would appear to be

$$y = Ae^{m_1 x} + Be^{m_2 x} = Ce^{m_1 x}$$

where $C = A + B$. The solution now only contains one constant of integration, and we know that it should contain two.

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In fact, in this case, the general solution is

$$\boxed{y = (A + Bx)e^{m_1 x}},$$

where A and B are arbitrary constants.

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In fact, in this case, the general solution is

$$\boxed{y = (A + Bx)e^{m_1 x}},$$

where A and B are arbitrary constants.

Example: Solve $\quad y'' - 6y' + 9y = 0$

$$y = Ae^{mx} \quad \Rightarrow \quad Ae^{mx}(m^2 - 6m + 9) = 0 \quad \Rightarrow \quad (m-3)^2 = 0.$$

So $m = 3$ is repeated and the general solution is $\quad y = (A + Bx)\, e^{3x}$.

3. Suppose the roots are

$$m_1 = \alpha + i\beta, \quad m_2 = \alpha - i\beta.$$

The general solution for y is now

$$\begin{aligned} y &= Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x} = Ae^{\alpha x}e^{i\beta x} + Be^{\alpha x}e^{-i\beta x} \\ &= e^{\alpha x}(Ae^{i\beta x} + Be^{-i\beta x}) = e^{\alpha x}((A+B)\cos(\beta x) + i(A-B)\sin(\beta x)) \\ &= \boxed{e^{\alpha x}\left(C\cos(\beta x) + D\sin(\beta x)\right)}, \end{aligned}$$

where C and D are arbitrary constants (they are, in fact, combinations of A and B.) This is the general solution in this case.

3.9.5 Example

Find the general solution of $\frac{d^2y}{dx^2} + y = 0$

3.9.5 Example

Find the general solution of $\dfrac{d^2y}{dx^2} + y = 0$

The auxiliary equation is $\quad m^2 + 1 = 0,$

so that the solutions are $m_1 = i$ and $m_2 = -i$; that is $\alpha = 0$, $\beta = 1$.

So the general solution in this case is $\quad y = C\cos x + D\sin x.$

The **GENERAL SOLUTION** of the ODE

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

is found by solving the **auxiliary equation** $am^2 + bm + c = 0$, then

1. $\boxed{y = Ae^{m_1 x} + Be^{m_2 x}}$
   if m_1 and m_2 are the **distinct** roots of the auxiliary equation
2. $\boxed{y = (A + Bx)\, e^{m_1 x}}$
   if $m_2 = m_1$ are **repeated** roots of the auxiliary equation
3. $\boxed{y = e^{\alpha x}\,(A\cos(\beta x) + B\sin(\beta x))}$
   if $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ are **complex conjugate** roots of the auxiliary equation.

In all cases, A and B are *arbitrary real constants* (for convenience, in the final case, we have reverted to A and B instead of C and D).

The linear ODE

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x)$$

is said to be **inhomogeneous**, i.e. there is a term involving x but not y on the RHS.

This equation is solved in two stages:

3.10 The inhomogeneous case

The linear ODE

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = r(x)$$

is said to be **inhomogeneous**, i.e. there is a term involving x but not y on the RHS.

This equation is solved in two stages:

1. Consider the equivalent homogeneous equation obtained by setting $r(x) = 0$,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

   Solve this using the previous method;
   This solution is called the **complementary function** (C.F.) written $u(x)$ say.

2. Obtain (for example by trial and error) a **particular solution** to the original inhomogeneous equation, or **particular integral** (P.I.), written $v(x)$ say.

   Note: Usually, the form of the P.I. is closely related to the function $r(x)$. We can use our experience of derivatives of functions to help find $v(x)$. For example differentiating

   - e^{px} gives pe^{px} (use if RHS is an exponential)
   - x^n gives nx^{n-1} (use if RHS is a polynomial)
   - $\sin px$ gives $p\cos px$ (use if RHS is $\sin / \cos$)
   - $\cos px$ gives $-p\sin px$ (use if RHS is $\sin / \cos$)

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   - x^n gives nx^{n-1} (use if RHS is a polynomial)
   - $\sin px$ gives $p \cos px$ (use if RHS is sin / cos)
   - $\cos px$ gives $-p \sin px$ (use if RHS is sin / cos)

   The complete general solution is then

$$\boxed{y = u(x) + v(x)}.$$

3.10.1 Example

Find the general solution of $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{-x}$.

3.10.1 Example

Find the general solution of $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^{-x}$.

- First solve the equivalent homogeneous equation ie set RHS to zero.

  Auxiliary equation is

  $$m^2 - 3m + 2 = 0$$

  with solutions $m = 1$, $m = 2$.

  Hence the complementary function is

  $$u(x) = Ae^{x} + Be^{2x}.$$

- Now seek a particular solution related to $r(x) = e^{-x}$.
  Try a solution of the form

  $$v(x) = Ce^{-x}$$

then

$$\frac{dv}{dx} = -Ce^{-x} \text{ and } \frac{d^2v}{dx^2} = Ce^{-x}.$$

Substitute into the ODE:

$$Ce^{-x} - 3\left(-Ce^{-x}\right) + 2Ce^{-x} = e^{-x}.$$

ie

$$6Ce^{-x} = e^{-x}.$$

Equating the coefficients of e^{-x} we obtain $\qquad C = 1/6.$
So the P.I. is

$$v(x) = \frac{1}{6}e^{-x}.$$

The general solution to the ODE is the C.F. plus the P.I.:

$$y = Ae^{x} + Be^{2x} + \frac{1}{6}e^{-x}.$$

3.10.2 Example

Find the general solution of $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 3 - 2x^2.$

3.10.2 Example

Find the general solution of $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 3 - 2x^2.$

Since the LHS is the same as example (9), the C.F. is the same.

- Now find the P.I.:

  since $\quad r(x) = 3 - 2x^2$, try

$$v(x) = ax^2 + bx + c.$$

  and find appropriate values for a, b and c.

  Now

$$\frac{dv}{dx} = 2ax + b, \qquad \frac{d^2v}{dx^2} = 2a.$$

- Substitute into the ODE:

$$2a - 3(2ax + b) + 2(ax^2 + bx + c) = 3 - 2x^2.$$

3.10.2 Example (ctd)

- Collecting terms

$$2ax^2 + x(-6a + 2b) + (2a - 3b + 2c) = 3 - 2x^2.$$

- Now find a, b and c by equating coefficients:
  x^2 : $\quad 2a = -2 \quad \Rightarrow \quad a = -1.$
  x^1 : $\quad -6a + 2b = 0 \quad \Rightarrow \quad b = -3.$
  x^0 : $\quad 2a - 3b + 2c = 3 \quad \Rightarrow \quad c = -2.$
  So the PI is $v(x) = -x^2 - 3x - 2.$
  So the complete general solution to the ODE is (CF+PI):

$$y = Ae^x + Be^{2x} - x^2 - 3x - 2.$$

3.10.3 Example

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \sin x.$$

Again the C.F. is the same as before.

- Try to find a P.I. of the form $v(x) = a \sin x$ and find an appropriate value of a.

  When we substitute into the ODE we obtain

  $$-a \sin x - 3a \cos x + 2a \sin x = \sin x.$$

- Collecting terms: $a \sin x - 3a \cos x = \sin x$
- Now equate coefficients of $\sin x$ and of $\cos x$: there is no value of a which will make both

  $$a \sin x = \sin x \quad \text{and} \quad -3a \cos x = 0.$$

  So need to try a different form for the PI.

3.10.3 Example (ctd)

- So, try a P.I. of the form
  $v(x) = a\sin x + b\cos x$ and find appropriate values for a and b.
- Substituting into the ODE we obtain

$$-a\sin x - b\cos x - 3a\cos x + 3b\sin x + 2a\sin x + 2b\cos x = \sin x.$$

- Collecting terms

$$(a + 3b)\sin x + (b - 3a)\cos x = \sin x.$$

- Now equate coefficients:

$$\begin{aligned} \cos x : \quad & b - 3a = 0 \;\Rightarrow\; b = 3a. \\ \sin x : \quad & a + 3b = 1 \;\Rightarrow\; a + 9a = 1. \end{aligned}$$

  Solving for a and b, $\quad a = 1/10, \; b = 3/10.$
  So the solution to the ODE is:

$$y = Ae^{x} + Be^{2x} + \frac{1}{10}\sin x + \frac{3}{10}\cos x.$$

3.10.4 Example

Note: this method may fail if the RHS is the same for as the CF.

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x.$$

This is an example of an *exceptional case:* the function $r(x) = e^x$ appears in the C.F. $u(x) = Ae^x + Be^{2x}$.

- In this case the natural first choice for the P.I.,

$$v(x) = ae^x,$$

fails, as it gives

$$ae^x - 3ae^x + 2ae^x = e^x \quad \Rightarrow \quad 0 = 1 \quad ??$$

3.10.4 Example (ctd)

- The solution is to try a P.I. of the form $v(x) = axe^x$.

  Then
  $$\frac{dv}{dx} = axe^x + ae^x, \qquad \frac{d^2v}{dx^2} = axe^x + 2ae^x.$$

- Substitute into the ODE:
  $$axe^x + 2ae^x - 3(axe^x + ae^x) + 2axe^x = e^x.$$

  Now all the terms in xe^x cancel out, leaving
  $$2ae^x - 3ae^x = e^x \quad \Rightarrow \quad a = -1,$$

  so the solution to the ODE is
  $$y = Ae^x + Be^{2x} - xe^x.$$

Similar modifications with other exceptional cases.

3.11 Boundary conditions and initial conditions

In general, the solution to an nth order ODE contains n arbitrary constants. To determine the values of these constants, n additional conditions must be specified. For a second-order equation two additional conditions are needed, and they can be either

- *Initial Conditions*
  If two conditions (usually y and y') are given at the same value of x (e.g. $x = 0$)

  or

- *Boundary Conditions*
  If one condition is given at $x = x_1$ (say) and the second condition is given at $x = x_2$ (say), or perhaps a limit as $x \to \infty$.

3.11.1 Example

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$$

with the initial conditions

$$y = 1 \quad \text{and} \quad \frac{dy}{dx} = 0 \quad \text{at} \quad x = 0.$$

The general solution to the ODE is

$$y = Ae^x + Be^{2x} - xe^x,$$

from the previous example.
At $x = 0$,

$$y = A + B = 1 \quad \text{and} \quad \frac{dy}{dx} = A + 2B - 1 = 0.$$

Hence $A = 1$ and $B = 0$, so the solution to the ODE with these initial conditions is

$$y = e^x - xe^x.$$

Find the curve in the (x, y) plane that passes through (0,3) and whose tangent line at a point (x, y) has slope $2x/y^2$.

3.12.1 Miscellaneous examples

Find the curve in the (x, y) plane that passes through (0,3) and whose tangent line at a point (x, y) has slope $2x/y^2$.

Since the slope of the curve $y = y(x)$ is $\frac{dy}{dx}$, we have $\dfrac{dy}{dx} = \dfrac{2x}{y^2}$

and since it must pass through (0,3) we have the condition $y(0) = 3$.

Separating the variables, we find $\displaystyle\int y^2 dy = \int 2x dx + c$

where c is an arbitrary constant. Hence $\frac{1}{3}y^3 = x^2 + c$.

$$y(0) = 3 \Rightarrow \frac{1}{3}3^3 = 0 + c \quad \text{so} \quad c = 9.$$

Finally

$$\frac{1}{3}y^3 = x^2 + 9 \quad \text{or} \quad y = \left(3x^2 + 27\right)^{\frac{1}{3}}.$$

3.12.2 The LCR circuit revisited

Recall that the ODE for the current $I(t)$ flowing around a circuit with an applied voltage $V(t)$ in the presence of a resistance R, capacitance C and inductance L is

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = \frac{dV}{dt}.$$

So if $V(t)$ is constant, this becomes

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = 0.$$

If we seek a solution $I(t) = Ae^{mt}$ then m satisfies the auxiliary equation

$$Lm^2 + Rm + \frac{1}{C} = 0.$$

This has solutions

$$m_1 = \frac{-R + \sqrt{R^2 - 4\frac{L}{C}}}{2L} \quad \text{and} \quad m_2 = \frac{-R - \sqrt{R^2 - 4\frac{L}{C}}}{2L}$$

If

$$R^2 - 4\frac{L}{C} > 0$$

then the solutions for m_1 and m_2 are real and the solutions are of exponential form and decay with time.
However, if

$$R^2 - 4\frac{L}{C} < 0$$

then the solutions are complex and we get oscillatory solutions of the form

$$I(t) = A\exp\left(-\frac{Rt}{2L}\right)\cos\left(\frac{t}{2L}\sqrt{R^2 - 4\frac{L}{C}}\right).$$

Solve the ODE $\quad y'' - y' - 2y = 0.$

3.12.3 Example

Solve the ODE $y'' - y' - 2y = 0.$

Solution: Seeking a solution of the form $y = Ae^{mx}$ gives that m satisfies the auxiliary equation

$$m^2 - m - 2 = 0.$$

$$\text{Hence} \quad (m-2)(m+1) = 0$$

so $m = 2$ and $m = -1$ are the two roots.
Hence the general solution is

$$y = Ae^{2x} + Be^{-x}$$

for arbitrary constants A and B.

3.12.4 Example

Solve the ODE $y'' + 4y' + 4y = 0$.

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Solution: Seeking a solution of the form $y = Ae^{mx}$ gives that m satisfies the auxiliary equation

$$m^2 + 4m + 4 = 0. \quad \text{Hence} \quad (m+2)^2 = 0$$

so $m = -2$ is a repeated root.
Hence the general solution is

$$y = (A + Bx)e^{-2x}$$

for arbitrary constants A and B.

3.12.5 Example

Solve the ODE $\quad y'' - 2y' + 5y = 0.$

3.12.5 Example

Solve the ODE $\quad y'' - 2y' + 5y = 0.$

Solution: Seeking a solution of the form $y = Ae^{mx}$ gives that m satisfies the auxiliary equation

$$m^2 - 2m + 5 = 0.$$

This cannot be factorised and so we solve it by formula to get $m = 1 + 2i$ and $m = 1 - 2i$ as the roots.
Hence the general solution is

$$y = e^x(A\cos 2x + B\sin 2x)$$

where A and B are arbitrary constants.