The zeros of certain differential polynomials

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Abstract

Let f be a function transcendental and meromorphic in the plane and let

$$\psi = a\left(f^{(m)} + \ldots + a_0 f\right)\left(f^{(k)} + \ldots + b_0 f\right)^n - 1,$$

where a and the coefficients a_j and b_j are meromorphic functions of small growth compared to f. Under appropriate conditions on the integers m, k and n, estimates are given for the frequency of zeros of ψ in terms of the growth of f.
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1 Introduction

The application of the Nevanlinna theory to the value distribution of meromorphic functions and their derivatives goes back to the work of Milloux and beyond [8, Chapter 3], and has been much influenced by the landmark paper [7] of Hayman. A conjecture of Hayman from [7] was settled by the following result of Bergweiler and Eremenko [2].

Theorem 1.1 ([2]) *Let f be transcendental and meromorphic in the plane, let n and k be integers with $n > k \geq 1$, and let b be a non-zero complex number. Then $(f^n)^{(k)} - b$ has infinitely many zeros.*

Partial results for the case $k = 1$ had been proved earlier by a number of authors [3, 7, 9]. The following conjecture was advanced in [13], and represents a natural generalisation of the case $n = 2, k = 1$ of Theorem 1.1.

Conjecture 1.1 ([13]) *If f is transcendental and meromorphic in the plane and k is a positive integer and b is a non-zero complex number, then $ff^{(k)} - b$ has infinitely many zeros.*

The example $f(z) = R(z)e^{P(z)}$, where R is a rational function and P is a non-constant polynomial, shows that $ff^{(k)}$ may have finitely many zeros: for $k \geq 2$ there are no other transcendental meromorphic functions in the plane with this property [4, 5, 11]. Obviously Theorem 1.1 shows that Conjecture 1.1 is true for $k = 1$, and when f is entire the conjecture has also been established for $k = 2$ [12]. Results related to Conjecture 1.1, concerning the value distribution of $f(f^{(k)})^n$ when $k \geq 1$ and $n \geq 2$, may be found in [13, 14].

The following notational conventions will be used throughout this paper. If f is a function transcendental and meromorphic in the plane then Λ_f will denote the set of all functions a meromorphic in the plane such that $T(r,a) = S(r,f)$, where $S(r,f)$ as usual denotes any quantity satisfying

$$S(r,f) = o(T(r,f)) \quad \text{as} \quad r \to \infty,$$

possibly outside a set of finite measure. It is well known that Λ_f is a field closed under differentiation. The following theorem was proved in [1].

Theorem 1.2 ([1]) *Let f be a function transcendental and meromorphic in the plane, and let*

$$\psi = afG^n - 1, \tag{1}$$

where $n \geq 2$ and

$$G = L[f], \quad L[w] = w^{(k)} + b_{k-1}w^{(k-1)} + \ldots + b_0 w, \quad k \in \mathbb{N}, \quad a, b_j \in \Lambda_f, \quad a \not\equiv 0. \tag{2}$$

Assume that $G \notin \Lambda_f$ but that $\phi \in \Lambda_f$ for every solution ϕ of $L[w] = 0$ which is meromorphic in the plane. Then

$$T(r,f) \leq d_0 \overline{N}(r, 1/\psi) + S(r,f), \tag{3}$$

where

$$d_0 = \frac{n(k+1)}{(1-\delta_k)(n-1)}, \quad \delta_k = \frac{n(k+1)}{(n-1)(1+n(k+1))} \in (0,1). \tag{4}$$

In particular, taking $G = f^{(k)}$ in (2) shows that if f is a transcendental meromorphic function and $k \geq 1$ and $n \geq 2$, then the differential monomial $f(f^{(k)})^n$ takes every finite non-zero value infinitely often. The first result of the present paper is a direct analogue of Theorem 1.2 but with f replaced by a linear differential polynomial F in (1).

Theorem 1.3 *Let f be a function transcendental and meromorphic in the plane. Let m, k and n be integers satisfying*

$$m \geq 0, \quad k \geq 1, \quad n \geq 2, \quad m + 1 < (n^2 - 2n)k + n^2 - n. \tag{5}$$

Define linear differential operators M and L by

$$M = D^m + \sum_{j=0}^{m-1} a_j D^j, \quad L = D^k + \sum_{j=0}^{k-1} b_j D^j, \quad D = \frac{d}{dz}, \quad a_j, b_j \in \Lambda_f, \tag{6}$$

where the operator M is the identity operator if $m = 0$. Assume that the homogeneous linear differential equations

$$M[w] = 0, \quad L[w] = 0, \tag{7}$$

have no common local solution other than the trivial solution $w = 0$. Define F, G and ψ by

$$F = M[f], \quad G = L[f], \quad \psi = aFG^n - 1, \quad a \in \Lambda_f, \quad a \not\equiv 0, \tag{8}$$

and assume that:
(a) neither F nor G belongs to Λ_f;

(b) if ϕ is meromorphic in the plane and satisfies $\phi = M[w]$ for some local solution w of $L[w] = 0$, then $\phi \in \Lambda_f$.

Then f satisfies (3), with

$$d_0 = \frac{s_0}{1-\eta}\left(\frac{n(k+1)+m}{n-1}\right), \tag{9}$$

where

$$s_0 = 1 + (mn+k-1)\left(\frac{n(k+1)+m+2}{n(k+1)+m+1}\right), \tag{10}$$

and

$$\eta = \frac{(k+m+1)n}{(n-1)(n(k+1)+m+1)} \tag{11}$$

satisfies $\eta \in (0,1)$. If $m = 0$ then s_0 may be replaced by 1 in (9).

A number of remarks are in order in connection with Theorem 1.3. First, an example in §11 will show that the hypothesis (b) cannot be deleted in Theorem 1.3.

Next, the assumption that the equations (7) have no non-trivial common local solution represents no real restriction in generality, since otherwise [6, Lemma D] there exist linear differential operators N, N_1, N_2 with coefficients in Λ_f such that $M = N_1 \circ N$ and $L = N_2 \circ N$, so that F, G and ψ may be regarded as differential polynomials in the meromorphic function $g = N[f]$: note, however, that it is then possible that $g \in \Lambda_f$, in which case $F, G, \psi \in \Lambda_f$, from which it follows at once that either $\psi \equiv 0$ or

$$\overline{N}(r, 1/\psi) = S(r, f). \tag{12}$$

Further, if $n = 2$ then the last condition of (5) forces $m = 0$ and so $F = f$ as in Theorem 1.2, while conversely if $m = 0$ then (5) is satisfied for any $n \geq 2$ and $k \geq 1$. Moreover, when $m = 0$ and consequently M is the identity operator, conditions (a) and (b) in Theorem 1.3 reduce to the hypotheses of Theorem 1.2, and in this case the constant d_0 in (9) agrees with that in (4).

Thus Theorem 1.3 is a direct generalisation of Theorem 1.2 to the case where f is replaced in (1) by a linear differential polynomial F. Nevertheless, the assumptions (a) and (b) are strong, and the following result addresses the question of what can be said when these hypotheses are weakened to the simple assumption that neither F nor G vanishes identically: for the proof of this result, however, the relatively strong hypothesis (12) on the frequency of zeros of ψ is required.

Theorem 1.4 *Let f be a function transcendental and meromorphic in the plane. Let m, k and n be integers satisfying (5), let the linear differential operators M and L be as in (6), and assume again that the homogeneous linear differential equations (7) have no non-trivial common local solution. Define F, G and ψ by (8) and assume that neither F nor G vanishes identically, and that (12) holds. Then at least one of the following is true.*
(i) The function f is a linear combination over $\mathbb{C}$ of local solutions of the equations (7).
(ii) The functions ψ'/ψ and G are in Λ_f, but ψ is not, and f, F and ψ satisfy

$$f = \alpha\psi + \beta, \quad F = \gamma(\psi + 1), \quad \alpha, \beta, \gamma \in \Lambda_f, \quad L[\alpha\psi] = 0, \quad a\gamma G^n = 1.$$

(iii) The functions ψ'/ψ and G'/G are in Λ_f, but ψ and G are not, and f, F and ψ satisfy

$$f = AG^{-n} + BG, \quad \psi = CG^{n+1}, \quad F = JG^{-n} + KG,$$

in which

$$A, B, C, J, K \in \Lambda_f, \quad L[AG^{-n}] = 0, \quad C = aK, \quad aJ = 1.$$

In all three cases

$$\overline{N}(r, f) = S(r, f). \tag{13}$$

Examples will be given in §11 to show that each of the conclusions (i), (ii) and (iii) may occur.

2 A lemma required for Theorems 1.3 and 1.4

The following fairly standard lemma is a refinement of [6, Lemma D].

Lemma 2.1 *Let s, t be non-negative integers and let S and T be linear differential operators over Λ_f, both not the zero operator, and of orders s and t respectively, such that the equations $S[w] = 0, T[w] = 0$, have no nontrivial common local solution. Then there exist linear differential operators U and V over Λ_f, of orders at most $\max\{t-1, 0\}$ and $\max\{s-1, 0\}$ respectively, such that*

$$1 = U \circ S + V \circ T,$$

in which 1 denotes the identity operator.

Proof. Assume without loss of generality that $s \geq t$. The lemma is then established in a standard way by induction on t, and is plainly true for $t = 0$ since in this case it is possible to take $U = 0$ and V of order 0. Assume now that $t = n$ is a positive integer and that the lemma holds when one of the operators has order less than n. By the division algorithm for linear differential operators [10, p.126] there exist operators T_1, T_2 such that T_1 has order $s - t$, while T_2 has order $u < t$, and

$$S = T_1 \circ T + T_2.$$

Moreover, the equations $T[w] = 0, T_2[w] = 0$, have no nontrivial common local solution since such a common local solution would also solve $S[w] = 0$. In particular, T_2 is not the zero operator. By the induction hypothesis there exist linear differential operators X and Y over Λ_f, such that

$$1 = X \circ T + Y \circ T_2 = X \circ T + Y \circ (S - T_1 \circ T).$$

Here X and Y both have orders at most $t - 1$, using the fact that $\max\{u - 1, 0\} \leq t - 1$. Set $U = Y$ and $V = X - Y \circ T_1$. Then U and V have orders at most $t - 1$ and

$$\max\{t - 1, s - t + t - 1\} \leq s - 1$$

respectively, thus completing the induction.

3 Proof of Theorems 1.3 and 1.4: first steps

Assume the common hypotheses of Theorems 1.3 and 1.4, that is, that f is transcendental and meromorphic in the plane, that the integers k, m and n satisfy (5), while F, G and ψ satisfy (6) and (8) with $FG \not\equiv 0$, and finally that the equations (7) have no non-trivial common local solution. By (6) and Lemma 1 of [6], there exist linear differential operators P and Q of orders k and m respectively, such that

$$\begin{aligned} P \circ M &= Q \circ L, \quad P = \sum_{q=0}^{k} \alpha_q D^q, \quad Q = \sum_{q=0}^{m} \gamma_q D^q, \\ H = P[F] = \sum_{q=0}^{k} \alpha_q F^{(q)} &= Q[G] = \sum_{q=0}^{m} \gamma_q G^{(q)}, \quad \alpha_q, \gamma_q \in \Lambda_f, \quad \alpha_k = \gamma_m = 1. \end{aligned} \tag{14}$$

Suppose first that $H \equiv 0$. Then

$$(P \circ M)[f] = (Q \circ L)[f] \equiv 0$$

by (8), and so f is a linear combination of local solutions of the homogeneous linear differential equations $M[w] = 0, L[w] = 0$. In the setting of Theorem 1.4 this is conclusion (i), in which case it is clear that (13) holds. Moreover, $F = M[f] = M[w]$ for some local solution w of $L[w] = 0$, which with the hypotheses of Theorem 1.3 gives an immediate contradiction, since on the one hand $F = M[w] \in \Lambda_f$ while on the other hand $F \notin \Lambda_f$ by assumption.

Assume henceforth that $H \not\equiv 0$, so that f is not a linear combination of local solutions of the homogeneous linear differential equations $M[w] = 0, L[w] = 0$. By Lemma 2.1 and the fact that the equations (7) have no non-trivial common local solution, there also exist linear differential operators R_1, R_2, each with coefficients in the field Λ_f, such that

$$1 = R_1 \circ M + R_2 \circ L, \quad f = R_1[F] + R_2[G], \tag{15}$$

in which 1 denotes the identity operator and the second relation follows from the first and (8). Moreover by (6), (8) and Lemma 2.1 the operators R_1, R_2 have orders at most $k-1, m-1$ respectively. This leads to the following estimate for the growth of f in terms of ψ.

Lemma 3.1 *The functions f and ψ satisfy*

$$T(r, f) \leq s_0 T(r, \psi) + S(r, f) \quad \textit{as} \quad r \to \infty, \tag{16}$$

where s_0 is given by (10). If $m = 0$ in (6) then (16) holds with $s_0 = 1$.

Proof. Define

$$\begin{aligned} \Phi &= \frac{afH^n}{\psi + 1} = \frac{fH^n}{FG^n} = \frac{R_1[F]}{F} \left(\frac{H}{G}\right)^n + \frac{R_2[G]}{G} \left(\frac{H}{G}\right)^{n-1} \frac{H}{F} \\ &= \frac{R_1[F]}{F} \left(\frac{Q[G]}{G}\right)^n + \frac{R_2[G]}{G} \left(\frac{Q[G]}{G}\right)^{n-1} \frac{P[F]}{F}, \end{aligned} \tag{17}$$

using (8), (14) and (15). Then $\Phi \not\equiv 0$, and the lemma of the logarithmic derivative gives

$$m(r,\Phi) = S(r,f). \tag{18}$$

Next, the operators R_1, R_2, P and Q in (17) each have orders at most $k-1, m-1, k$ and m respectively, and the logarithmic derivatives $F^{(j)}/F, G^{(j)}/G$ each have poles of multiplicity at most j. Since the coefficients of P, Q, R_1 and R_2 are in Λ_f this implies that

$$N(r,\Phi) \leq (mn+k-1)\overline{N}(r,\Phi) + S(r,f). \tag{19}$$

But poles of Φ can only arise from zeros of $\psi + 1$ or poles of a, f or the coefficients of $P \circ M$, since $H = (P \circ M)[f]$. Moreover, a pole of f which is not a zero of a nor a pole of any of the coefficients a_j, b_j is by (6) and (8) a pole of ψ of multiplicity at least $n(k+1)+m+1$, so that

$$\overline{N}(r,f) \leq \left(\frac{1}{n(k+1)+m+1}\right) T(r,\psi) + S(r,f). \tag{20}$$

Combining (18), (19) and (20) gives

$$\begin{aligned} T(r,\Phi) &\leq (mn+k-1)(\overline{N}(r,f) + \overline{N}(r,1/(\psi+1))) + S(r,f) \\ &\leq (mn+k-1)\left(\frac{n(k+1)+m+2}{n(k+1)+m+1}\right) T(r,\psi) + S(r,f). \end{aligned} \tag{21}$$

The definition (17) of Φ leads at once to

$$\frac{1}{f} = \frac{aH^n}{\Phi(\psi+1)}, \quad \frac{1}{f^{n+1}} = \frac{H^n}{f^n} \cdot \frac{a}{\Phi(\psi+1)}. \tag{22}$$

The second relation of (22) then yields

$$m(r,1/f) \leq m(r,1/f^{n+1}) \leq m(r,1/\Phi) + m(r,1/(\psi+1)) + S(r,f),$$

while the first gives

$$N(r,1/f) \leq N(r,1/\Phi) + N(r,1/(\psi+1)) + S(r,f),$$

and combining the last two estimates with (21) proves (16). This completes the proof of Lemma 3.1, the last assertion holding since $m = 0$ gives $F = f$, $H = G$ and $\Phi = 1$.

4 Continuation of the proofs of Theorems 1.3 and 1.4

Differentiating (14) gives

$$H' = \frac{H'}{H}P[F] = P[F]' = \sum_{q=0}^{k+1} \beta_q F^{(q)}, \quad \beta_q = \alpha_q' + \alpha_{q-1}, \quad \alpha_{-1} = \alpha_{k+1} = 0, \quad \beta_{k+1} = 1. \tag{23}$$

Hence (14) and (23) lead to a linear differential equation

$$\sum_{j=0}^{k+1} \left(\beta_j - \frac{H'}{H}\alpha_j\right) w^{(j)} = 0, \tag{24}$$

satisfied by $w = F$. Now write

$$F = uv, \quad v = \frac{1}{aG^n}, \quad c_j = \beta_j - \frac{H'}{H}\alpha_j. \tag{25}$$

Using the standard convention, for integers j and q,

$$\binom{j}{q} = \frac{j!}{q!(j-q)!} \quad (0 \leq q \leq j), \quad \binom{j}{q} = 0 \quad \text{(otherwise)},$$

substituting (25) into (24) yields

$$0 = \sum_{j=0}^{k+1} c_j F^{(j)} = \sum_{j=0}^{k+1} c_j \sum_{q=0}^{k+1} \binom{j}{q} u^{(q)} v^{(j-q)}. \tag{26}$$

Dividing by v and reversing the order of summation in (26) now gives

$$0 = \sum_{q=0}^{k+1} u^{(q)} A_q, \quad A_q = \sum_{j=0}^{k+1} \binom{j}{q} c_j \frac{v^{(j-q)}}{v}. \tag{27}$$

In particular, using (23) and (25),

$$A_{k+1} = c_{k+1} = 1, \quad A_0 = \sum_{j=0}^{k+1} c_j \frac{v^{(j)}}{v} = \sum_{j=0}^{k+1} \left(\beta_j - \frac{H'}{H}\alpha_j \right) \frac{v^{(j)}}{v}. \tag{28}$$

Two cases will now be considered, depending on whether or not A_0 vanishes identically.

5 Case 1: suppose that $A_0 \not\equiv 0$

From (8), (25) and (27),

$$u = \frac{F}{v} = aFG^n = \psi + 1, \quad \sum_{q=0}^{k+1} \psi^{(q)} A_q = -A_0,$$

and hence

$$-\frac{1}{\psi} = 1 + \frac{1}{A_0} \sum_{q=1}^{k+1} B_q, \quad B_q = A_q \frac{\psi^{(q)}}{\psi}. \tag{29}$$

By (25), (27) and (29),

$$m(r, A_q) + m(r, B_q) = S(r, f), \quad m(r, 1/\psi) \leq m(r, 1/A_0) + S(r, f). \tag{30}$$

Next, (25), (27) and (29) give, for $q = 0, \ldots, k+1$,

$$B_q = \sum_{j=0}^{k+1} \binom{j}{q} \left(\beta_j - \frac{H'}{H}\alpha_j \right) \frac{v^{(j-q)}}{v} \frac{\psi^{(q)}}{\psi}, \quad v = \frac{1}{aG^n}.$$

In particular, poles of B_q can only arise from poles of a, poles of the β_j and α_j, poles of the coefficients of F, G or H, zeros of a or G, zeros of H which are not zeros of G, zeros of ψ or poles of f. Moreover, since $\alpha_{k+1}=0$ by (23), the total contribution to $n(r,B_q)$ from the terms $H'/H, v^{(j-q)}/v, \psi^{(q)}/\psi$ is at most $k+1$. Hence (29) and (30) give

$$\begin{aligned} T(r,\psi) &\leq m(r,1/\psi)+N(r,1/\psi)+O(1) \\ &\leq m(r,1/A_0)+N(r,1/A_0)+(k+1)\overline{N}(r,1/\psi)+S(r,f) \\ &\leq T(r,A_0)+(k+1)\overline{N}(r,1/\psi)+S(r,f) \\ &\leq N(r,A_0)+(k+1)\overline{N}(r,1/\psi)+S(r,f). \end{aligned} \tag{31}$$

The same analysis leads to

$$N(r,A_0) \leq (k+1)\left(\overline{N}(r,f)+\overline{N}(r,1/G)\right)+\overline{N}(r,G/H)+S(r,f). \tag{32}$$

Since (14) gives

$$\overline{N}(r,G/H) \leq T(r,H/G)+O(1) \leq m\left(\overline{N}(r,f)+\overline{N}(r,1/G)\right)+S(r,f),$$

(31) and (32) combine to yield

$$T(r,\psi) \leq (k+m+1)\left(\overline{N}(r,f)+\overline{N}(r,1/G)\right)+(k+1)\overline{N}(r,1/\psi)+S(r,f). \tag{33}$$

But (6) and (8) imply that a zero of G which is not a pole of a nor of any of the a_j or b_j is a zero of $\psi+1$, and a zero of ψ' of multiplicity at least $n-1$, so that

$$\begin{aligned} \overline{N}(r,1/G) &\leq \frac{1}{n-1}N(r,\psi/\psi')+S(r,f) \leq \frac{1}{n-1}T(r,\psi'/\psi)+S(r,f) \\ &\leq \frac{1}{n-1}N(r,\psi'/\psi)+S(r,f) \leq \frac{1}{n-1}\left(\overline{N}(r,f)+\overline{N}(r,1/\psi)\right)+S(r,f), \end{aligned}$$

which on combination with (33) gives

$$\begin{aligned} T(r,\psi) &\leq (k+m+1)\left(1+\frac{1}{n-1}\right)\overline{N}(r,f)+\left(k+1+\frac{k+m+1}{n-1}\right)\overline{N}(r,1/\psi)+S(r,f) \\ &= \left(\frac{(k+m+1)n}{n-1}\right)\overline{N}(r,f)+\left(\frac{n(k+1)+m}{n-1}\right)\overline{N}(r,1/\psi)+S(r,f). \end{aligned} \tag{34}$$

Substituting (20) into (34) leads to

$$(1-\eta)\,T(r,\psi) \leq \left(\frac{n(k+1)+m}{n-1}\right)\overline{N}(r,1/\psi)+S(r,f), \tag{35}$$

where η is given by (11). But adding $nk+(n-1)(m+1)$ to both sides of (5) yields

$$nk+n(m+1) < (n-1)(m+1)+(n^2-n)(k+1), \quad \eta=\frac{(k+m+1)n}{(n-1)(n(k+1)+m+1)}<1.$$

Thus (16) and (35) together give (3) and (9) as in the conclusion of Theorem 1.3. Moreover, (3) and (12) are incompatible and so, with the hypotheses of Theorem 1.4, Case 1 is impossible.

To complete the proofs of Theorems 1.3 and 1.4, it remains to prove that the case $A_0 \equiv 0$ cannot arise with the hypotheses of Theorem 1.3, and in the setting of Theorem 1.4 to show that $A_0 \equiv 0$ leads to either conclusion (ii) or (iii).

6 Case 2: suppose that $A_0 \equiv 0$

Then (14), (23) and (28) give $d \in \mathbb{C}$ with

$$P[v]' = \frac{H'}{H} P[v], \quad P[v] = dH = dQ[G] = dP[F]. \tag{36}$$

It follows using (6), (8), (14) and (25) that any zero of G which is not a pole of a nor of any of the coefficients of P, M or L is a pole of $P[v]$ but not of f or $P[F]$. Hence

$$\overline{N}(r, 1/G) = S(r, f). \tag{37}$$

Similar considerations give (13) if $d \neq 0$. But if $d = 0$ then a zero of v which is not a pole of any of the coefficients of P cannot have multiplicity greater than $k-1$, by the existence-uniqueness theorem for solutions of linear differential equations. Since a pole of f which is not a pole of a nor of any of the coefficients of L is a zero of v of multiplicity at least $n(k+1) > k$, by (8) and (25), it follows that (13) holds, whether or not $d = 0$. Combining (13) and (37) then gives

$$G^{(j)}/G \in \Lambda_f \quad \text{for} \quad j \in \mathbb{N}. \tag{38}$$

7 Completion of the proof of Theorem 1.3

Assume the hypotheses (a) and (b) of Theorem 1.3 and that $A_0 \equiv 0$. The equation (36) gives

$$P[v - dF] = 0.$$

Write $v - dF = M[W]$ with W analytic on a suitably chosen small disc Ω. It follows at once that $(P \circ M)[W] = 0$ on Ω, so that W is a linear combination over $\mathbb{C}$ of local solutions of $L[w] = 0, M[w] = 0$, by the choice of P and Q in (14). Hence $v - dF = M[w]$ for some local solution of $L[w] = 0$, which implies by hypothesis (b) of Theorem 1.3 that $v - dF = \lambda \in \Lambda_f$, and hence $d \neq 0$, using (25), since $G \notin \Lambda_f$ by assumption. But this gives, using (15), (25) and (38),

$$F = \lambda_1 G^{-n} + \lambda_2, \quad f = R_1[F] + R_2[G] = \lambda_3 G^{-n} + \lambda_4 + \lambda_5 G, \tag{39}$$

using λ_j to denote elements of Λ_f. Hence

$$F = M[f] = \lambda_6 G^{-n} + \lambda_7 + \lambda_8 G, \quad \lambda_8 G = M[\lambda_5 G],$$

and since $G \notin \Lambda_f$ combining this relation with (39) shows that $\lambda_8 G = M[\lambda_5 G] = 0$. Moreover (38) and (39) now yield

$$G = L[f] = \lambda_9 G^{-n} + \lambda_{10} + \lambda_{11} G, \quad \lambda_9 G^{-n} = L[\lambda_3 G^{-n}], \quad \lambda_{10} = L[\lambda_4].$$

Again since $G \notin \Lambda_f$ it follows that $\lambda_9 G^{-n} = L[\lambda_3 G^{-n}] = 0$ and $\lambda_{10} = L[\lambda_4] = 0$. But then f is by (39) a linear combination over $\mathbb{C}$ of local solutions of $L[w] = 0, M[w] = 0$, and so $H \equiv 0$ in (14), which contradicts the assumption made following (14) that H does not vanish identically. This completes the proof of Theorem 1.3.

8 Completion of the proof of Theorem 1.4

Assume the hypothesis (12) of Theorem 1.4 and that $A_0 \equiv 0$. Then (12), (13) and (38) lead immediately to

$$\psi^{(j)}/\psi \in \Lambda_f \quad \text{and} \quad G^{(j)}/G \in \Lambda_f \quad \text{for} \quad j \in \mathbb{N}. \tag{40}$$

At this point it becomes necessary to divide the proof into two further subcases, depending on whether or not $G \in \Lambda_f$. For convenience the terms A_j, B_j, C_j, D_j, E_j will be used to denote elements of the field Λ_f.

9 Subcase 2A: suppose that $G \in \Lambda_f$

In this case (8), (15) and (40) make it possible to write

$$F = \gamma(\psi + 1), \quad f = \alpha\psi + \beta, \quad G = L[\alpha\psi] + L[\beta] = A_1\psi + L[\beta] = A_1\psi + A_2, \tag{41}$$

with $\alpha, \beta, \gamma \in \Lambda_f$ and $a\gamma G^n = 1$. Since $\psi \notin \Lambda_f$ by (16), it follows at once that $L[\alpha\psi] = A_1\psi = 0$ in (41). This gives conclusion (ii) of Theorem 1.4.

10 Subcase 2B: suppose that $G \notin \Lambda_f$

Combining (8), (15) and (40) leads to

$$F = a^{-1}G^{-n}(\psi + 1), \quad f = R_1[F] + R_2[G] = B_1\psi G^{-n} + B_2 G^{-n} + B_3 G. \tag{42}$$

To handle this case will require the following lemma.

Lemma 10.1 *With the assumptions of this subcase, suppose that $\psi G^{-n}, G^{-n}$ and G are linearly dependent over Λ_f. Then $\psi G^{-n-1} \in \Lambda_f$, and f satisfies conclusion (iii) of Theorem 1.4.*

Proof. Assume an identity

$$C_1\psi G^{-n} + C_2 G^{-n} + C_3 G = 0 \tag{43}$$

with the C_j not all identically zero. Then $C_1C_3 \not\equiv 0$, by (16) and the assumption that $G \notin \Lambda_f$. If $C_2 \equiv 0$ then the first conclusion of Lemma 10.1 follows at once, so assume that $C_2 \not\equiv 0$. Hence the relation (43) may be written in the form

$$C_4\psi = C_5 G^{n+1} + 1, \quad C_4C_5 \not\equiv 0. \tag{44}$$

Differentiating (44) now gives

$$\left(C_4' + \frac{\psi'}{\psi}C_4\right)\psi = \left(C_5' + (n+1)\frac{G'}{G}C_5\right)G^{n+1},$$

from which the first conclusion $\psi G^{-n-1} \in \Lambda_f$ of Lemma 10.1 follows at once using (40), unless

$$C_4' + \frac{\psi'}{\psi}C_4 = 0. \tag{45}$$

But (45) implies that ψC_4 is constant and $\psi \in \Lambda_f$, which contradicts (16).

Hence $\psi G^{-n-1} \in \Lambda_f$, and in view of the second relation in (42) it now follows that

$$f = AG^{-n} + BG, \quad \psi = CG^{n+1}, \quad A, B, C \in \Lambda_f.$$

Using (8), (40) and the fact that $G \notin \Lambda_f$ then gives

$$G = L[f] = D_1 G^{-n} + D_2 G, \quad D_1 G^{-n} = L[AG^{-n}] = 0, \quad F = M[f] = JG^{-n} + KG,$$

with $J, K \in \Lambda_f$, and using (8) again now leads to

$$CG^{n+1} = \psi = aKG^{n+1} + aJ - 1,$$

so that $aJ = 1$, $aK = C$, and f satisfies conclusion (iii) of Theorem 1.4. The proof of Lemma 10.1 is now complete.

To complete the proof of Theorem 1.4 in Subcase 2B, observe first that it follows from (42), on recalling (8) and using (40) again, that

$$G = E_1 \psi G^{-n} + E_2 G^{-n} + E_3 G, \quad E_1 \psi G^{-n} = L[B_1 \psi G^{-n}], \quad E_2 G^{-n} = L[B_2 G^{-n}]. \tag{46}$$

If E_1, E_2 are not both identically zero then the functions $\psi G^{-n}, G^{-n}$ and G are linearly dependent over Λ_f, and Lemma 10.1 shows that f satisfies conclusion (iii) of Theorem 1.4. Assume henceforth that E_1 and E_2 both vanish identically. Then (42) and (46) give

$$f = w_1 + B_3 G, \quad w_1 = B_1 \psi G^{-n} + B_2 G^{-n}, \quad L[w_1] = 0, \tag{47}$$

and, using (8) and (40),

$$F = M[f] = M[w_1] + M[B_3 G] = E_4 \psi G^{-n} + E_5 G^{-n} + E_6 G, \quad E_6 G = M[B_3 G]. \tag{48}$$

There are now two possibilities. If E_6 does not vanish identically in (48) then (42) and (48) imply that the functions $\psi G^{-n}, G^{-n}$ and G are linearly dependent over Λ_f, and Lemma 10.1 again shows that f satisfies conclusion (iii) of Theorem 1.4. On the other hand if E_6 vanishes identically in (48) then $M[B_3 G] = 0$ so that f satisfies conclusion (i) of Theorem 1.4 by (47), which contradicts the assumption made following (14) that $H = (P \circ M)[f] = (Q \circ L)[f]$ does not vanish identically. This completes the proof of Theorem 1.4.

11 Examples

In all six examples let $m = k = 1$ and $n = 4$. Then (5) is satisfied.

(I) Let

$$f(z) = e^{4z} + e^{-z} + 1, \quad F(z) = f'(z) + f(z) = 5e^{4z} + 1, \quad G(z) = f''(z) - 4f'(z) = 5e^{-z}.$$

Then

$$\psi(z) = \frac{1}{3125} F(z) G(z)^4 - 1 = \frac{e^{-4z}}{5}$$

has no zeros, and this example satisfies conclusion (i) of Theorem 1.4.

Moreover, in this case $w = e^{4z}$ is a solution of $L[w] = w'' - 4w' = 0$, and $\phi = M[w] = w' + w = 5e^{4z}$ is not in Λ_f, so that the hypothesis (b) in Theorem 1.3 is not redundant.

(II) This example again satisfies conclusion (i) of Theorem 1.4, but with ψ constant. Here $f(z) = e^{4z} + e^{-z}$ and

$$F(z) = f'(z)+f(z) = 5e^{4z}, \quad G(z) = f'(z)-4f(z) = -5e^{-z}, \quad \psi(z) = F(z)G(z)^4-1 = 3124.$$

(III) To obtain an example satisfying conclusion (ii) of Theorem 1.4, let

$$f(z) = e^z + z, \quad G(z) = f'(z) - f(z) = 1 - z, \quad F(z) = f'(z) = e^z + 1,$$

and

$$a(z) = (1-z)^{-4}, \quad \psi(z) = a(z)F(z)G(z)^4 - 1 = e^z.$$

(IV) A slightly more complicated example shows that conclusions (i) and (ii) of Theorem 1.4 may occur simultaneously. Let F, G, ψ be given by (8), where

$$f(z) = e^z + z + 1, \quad M = D = \frac{d}{dz}, \quad L = \left(D + \frac{1}{1-z}\right) \circ (D-1), \quad a(z) = (z-1)^4.$$

Since $M[1] = 0$ and $L[e^z + z] = 0$ it is clear that f satisfies conclusion (i). But

$$F(z) = f'(z) = e^z + 1, \quad G(z) = \left(D + \frac{1}{1-z}\right)[-z] = \frac{1}{z-1}, \quad \psi(z) = F(z) - 1 = e^z,$$

and so conclusion (ii) is also satisfied.

(V) Let

$$f(z) = e^{-4z} + e^z, \quad G(z) = f'(z) + 4f(z) = 5e^z, \quad F(z) = f'(z) = -4e^{-4z} + e^z,$$

and

$$a(z) = -\frac{1}{2500}, \quad \psi(z) = a(z)F(z)G(z)^4 - 1 = -\frac{e^{5z}}{4}.$$

This example satisfies conclusion (iii) of Theorem 1.4.

(VI) For an example which satisfies conclusions (i) and (iii) of Theorem 1.4, let F, G, ψ again be given by (8), with this time

$$f(z) = e^{-4z} + ze^z, \quad M = \left(D - \frac{z+2}{z+1}\right), \quad L = (D-1) \circ (D+4), \quad a(z) = \frac{-(z+1)}{625(5z+6)}.$$

Here $L[e^{-4z}] = L[e^z] = 0$ and $M[(z+1)e^z] = 0$ so that conclusion (i) is satisfied. Moreover,

$$F(z) = -\left(\frac{5z+6}{z+1}\right)e^{-4z} + \frac{e^z}{z+1}, \quad G(z) = (D-1)[(5z+1)e^z] = 5e^z, \quad \psi(z) = \frac{-e^{5z}}{5z+6},$$

and so f also satisfies conclusion (iii).

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