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Zeros of derivatives of meromorphic functions

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Abstract. The first part of this paper is an expanded version of a plenary lecture of the same title, given by the author at the CMFT conference at Bilkent University, Ankara, in June 2009. In the second part of the paper, a considerably stronger version of one of the main results is proved.

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1. The Gol'dberg conjecture

The conjecture of A.A. Gol'dberg concerns the link between the poles of a meromorphic function and the zeros of its derivatives, which has been the target of extensive research going back at least as far as Pólya [50]. Simple examples such as $f(z) = \tan z$ show that f may have infinitely many poles while the first derivative has no zeros, and similar examples arise as quotients of linearly independent solutions of the differential equation $w'' + Aw = 0$, where A is entire [34, Chapter 6]. Gol'dberg's conjecture involves derivatives of at least second order.

Conjecture 1.1 (The Gol'dberg conjecture)**.** *Let the function f be transcendental and meromorphic in $\mathbb{C}$, and let $k \geq 2$. Then*

$$\overline{N}(r,f) \leq N(r,1/f^{(k)}) + o(T(r,f)) \quad (n.e.).$$

Here we are using the standard terminology of Nevanlinna theory [17, 20, 45], in which $T(r,f)$ denotes the Nevanlinna characteristic, while $\overline{N}(r,f)$ counts the *distinct* poles of f and $N(r,1/f^{(k)})$ counts the poles of $1/f^{(k)}$, which of course are zeros of $f^{(k)}$, this time *with* multiplicity. Finally (n.e.) ("nearly everywhere") means as $r \to \infty$ outside a set of finite measure.

Thus the Gol'dberg conjecture asserts that the frequency of distinct poles of f is controlled by the zeros of $f^{(k)}$, up to an error $o(T(r,f))$. For example, $f(z) = \tan z$ satisfies

$$\overline{N}(r,f) \sim N(r,1/f'') \sim T(r,f) \sim \frac{2r}{\pi}.$$

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Some support for the Gol'dberg conjecture is lent by:

Theorem 1.1 (Part of the Pólya shire theorem [20, 50]). *Let the function f be meromorphic in $\mathbb{C}$ with at least two distinct poles. Let $w \in \mathbb{C}$ be such that the nearest pole of f to w is not unique. Then $f^{(k)}$ has a zero near w for all sufficiently large k.*

For $f(z) = \tan z$ the poles are at $\pi/2 + n\pi, n \in \mathbb{Z}$, and each pole is the centre of a "shire", an open vertical strip of width π. Near each point on the boundary lines there are zeros of $f^{(k)}$ for k sufficiently large.

The Pólya shire theorem implies that if f has at least two poles then $f^{(k)}$ has at least one zero for all large k, but it gives no information on the frequency of zeros of any specific derivative.

2. Further results supporting the Gol'dberg conjecture

The following theorem is a combination of results of Frank, Steinmetz and Weissenborn [12, 16, 56].

Theorem 2.1 ([12, 16, 56]). *Let the function f be transcendental and meromorphic in $\mathbb{C}$, let $k \geq 2$ and write*

$$L(w) = w^{(k)} + a_{k-1}w^{(k-1)} + \ldots + a_0 w, \tag{1}$$

where the coefficients a_j are rational functions. Then either f is a rational function in solutions of the differential equation $L(w) = 0$, or

$$k\overline{N}(r, f) \leq N(r, f) + N(r, 1/L(f)) + o(T(r, f)) \quad (n.e.).$$

See [7, 13, 36, 55] for further results on the zeros of linear differential polynomials $L(f)$. If we take all a_j to be 0 then $L(f) = f^{(k)}$ and a transcendental function f cannot be a rational function in solutions of $L(w) = 0$. Thus the Gol'dberg conjecture holds if all poles have multiplicity at most $k-1$, because a pole of f is then counted at most $k-1$ times in $N(r, f)$. In particular this is the case for the examples mentioned prior to Conjecture 1.1, since these have only simple poles. The conjecture is also true if the multiplicities of the poles tend to infinity since we then have

$$\overline{N}(r, f) = o(N(r, f)) = o(T(r, f)).$$

The next result is due to Frank [11, 14] for $n \geq m+3$ and to the author [35] for $n = m+2$.

Theorem 2.2 ([11, 14, 35]). *Let the function f be meromorphic in $\mathbb{C}$. If $f^{(m)}$ and $f^{(n)}$ have finitely many zeros, where $0 \leq m \leq n-2$, then f has finitely many poles and finite order, that is*

$$\rho(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r} < \infty, \tag{2}$$

and $f^{(m)} = Re^P$ with R a rational function and P a polynomial.

This result was conjectured by Hayman in 1959 [19], and related theorems of Hayman, Clunie and others concerning the zeros of derivatives of entire functions may be found in [20, Chapter 3]. The simple counterexample $f(z) = 1/(e^z + 1)$ shows that the result fails for $m = 0$, $n = 1$. For $n \geq m+3$ the proof of Theorem 2.2 uses Nevanlinna theory and a method of Frank involving properties of the Wronskian determinant [11, 13, 14], while the techniques for $n = m + 2$ in [35] are more asymptotic in character. The next result considers only one derivative, but requires a growth restriction on f.

Theorem 2.3 ([37]). *Let the function f be meromorphic of finite order in the plane. If $f^{(k)}$ has finitely many zeros, for some $k \geq 2$, then f has finitely many poles and $f^{(k)} = Re^P$ with R a rational function and P a polynomial.*

This result is again false for $k = 1$, since we may take $f(z) = \tan z$. Examples constructed in [37] show that Theorem 2.3 fails for infinite order, but there is a partial result [40] for finite *lower* order, that is if (2) holds with lim sup replaced by lim inf, provided the multiplicities of the poles do not grow too fast.

3. Linear differential operators

The formulation of Theorem 2.1 suggests replacing $f^{(k)}$ in Theorem 2.3 by $L(f)$, as defined by (1) with $k \geq 2$ and appropriate coefficients a_j. Examples such as

$$f(z) = \frac{1}{1 - e^z}, \quad f''(z) - f'(z) = \frac{2e^{2z}}{(1 - e^z)^3},$$

prompt the following conjecture.

Conjecture 3.1. *If the function f is meromorphic of finite order in $\mathbb{C}$ and $L(f)$ has finitely many zeros, where $k \geq 2$, $a_j \in \mathbb{C}$, then $\overline{N}(r, f) = \mathcal{O}(r)$ as $r \to \infty$.*

There is a partial result in [40], in which it is shown that $\overline{N}(r, f) = \mathcal{O}(r^3)$ unless all roots of the auxiliary equation

$$x^k + a_{k-1}x^{k-1} + \ldots + a_0 = 0$$

are distinct and collinear. This result seems highly unlikely to be sharp.

4. The Mues conjecture

Hayman observed in [19] that the derivative of a transcendental meromorphic function f in the plane takes every finite value, with at most one exception, infinitely often. It was conjectured by Mues [44] that the Nevanlinna deficiencies of the derivatives satisfy

$$\sum_{a\in\mathbb{C}} \delta(a, f^{(m)}) \leq 1$$

for $m \geq 1$, and this conjecture would follow from a positive resolution of the Gol'dberg conjecture. For partial results in the direction of the Mues conjecture see [30, 44, 58, 59, 60].

5. The Wiman conjecture

A meromorphic function on the plane is called real if it maps $\mathbb{R}$ into $\mathbb{R} \cup \{\infty\}$. The *Laguerre-Pólya class LP* consists of all entire functions f with the following property: there exist real polynomials P_n with only real zeros, such that $P_n \to f$ locally uniformly on $\mathbb{C}$. An example is

$$\exp(-z^2) = \lim_{n\to\infty} (1 - z^2/n)^n.$$

The Laguerre-Pólya theorem [33, 48] shows that f belongs to LP if and only if $f(z) = g(z)e^{-az^2}$ where $a \geq 0$ and g is a real entire function with real zeros and genus at most 1.

It is a simple consequence of Rolle's theorem and Hurwitz' theorem that if the function $f \in LP$ is transcendental and $k \geq 0$ then $f^{(k)}$ has only real zeros. The converse assertion was conjectured by Pólya [49]: if f is real entire and $f^{(k)}$ has only real zeros for all $k \geq 0$ then $f \in LP$. A much stronger conjecture was advanced by Wiman [1, 2]: for a real entire function f to belong to LP it suffices that f and f'' have only real zeros. We identify a number of milestones in the proof of these conjectures.

(i) (Levin and Ostrovskii 1960 [43]**):** *if f is a real entire function and f and f'' have only real zeros, then*

$$\log T(r, f) = \mathcal{O}(r \log r) \quad \text{as} \quad r \to \infty.$$

The paper [43] introduced a factorisation of the logarithmic derivative and the use of the Tsuji characteristic [17, 57], techniques which have played a fundamental role in most subsequent work on this subject.

(ii) (Hellerstein and Williamson 1977 [22, 23]**):** *if the function f is real entire, and all zeros of f, f' and f'' are real, then $f \in LP$* (thus proving Pólya's conjecture).

(iii) (Sheil-Small 1989 [54]**):** *Wiman's conjecture is true for f of finite order, that is, if (2) holds.*

(iv) (Bergweiler, Eremenko and Langley 2003 [5]**):** *Wiman's conjecture is true for f of infinite order* (thus filling the gap between (i) and (iii)).

Turning to higher derivatives, let the real entire function f have finitely many non-real zeros, and write $f = Ph$, where P is a real polynomial and h is real entire with real zeros.

(v) (Craven, Csordas and Smith 1987 [8, 9]**, Kim 1990** [32]**, Ki and Kim 2000** [31]**):** *if $h \in LP$ then $f^{(k)}$ has only real zeros, for all sufficiently large k.*

(vi) (Edwards and Hellerstein 2002 [10]**):** *if $h \notin LP$ and h has finite order then $f^{(k)}$ has at least two non-real zeros for each $k \geq 2$.*

(vii) (Bergweiler and Eremenko 2006 [4]**):** *if $h \notin LP$ and h has finite order then the number of non-real zeros of $f^{(k)}$ tends to ∞ with k.*

(viii) (Langley 2005 [38]**):** *if h has infinite order, then $f^{(k)}$ has infinitely many non-real zeros for $k \geq 3$ (and for $k = 2$ by (iv)).*

Combining (v), (vii) and (viii) leads to the following striking result: *for a real entire function f, the number of non-real zeros of the kth derivative $f^{(k)}$ tends to a limit as $k \to \infty$, and this limit is always either* 0 *or* ∞. In particular, this proves a conjecture of Pólya from 1943 [51].

The author's Ph.D. student D.A. Nicks has several results related to the Wiman conjecture, including the following [46]. *Let f be a real entire function, let $M > 0$ and $1 \leq j < k$. Suppose that the non-real zeros of $ff^{(j)}f^{(k)}$ have finite exponent of convergence and are zeros of f of multiplicity at least k but at most M. Then $f \in LP$ (and so all zeros of all $f^{(m)}$, $m \geq 0$, are real).*

5.1. Some key ideas for the proof of the Wiman conjecture. Suppose that f is a real entire function, and that f and f'' have only real zeros. Let

$$L(z) = \frac{f'(z)}{f(z)}, \quad F(z) = z - \frac{f(z)}{f'(z)}, \quad F' = \frac{ff''}{f'^2}.$$

Here the Newton function F has no critical values in $\mathbb{C} \setminus \mathbb{R}$. It is then possible to prove, using the Tsuji characteristic, the Levin-Ostrovskii factorisation of L and normal families, that F also has no *asymptotic* values in $\mathbb{C} \setminus \mathbb{R}$. If the upper half-plane is denoted by $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ then one of Sheil-Small's decisive insights in [54] was that

$$Y = \{z \in H : L(z) \in H\} \subseteq W = \{z \in H : F(z) \in H\}. \tag{3}$$

Moreover, the closure of Y contains no pole of L, since L has real poles with positive residues [54]. If $f \notin LP$ then it is possible to show [5, 54] that there exists a zero b of L on the boundary of a component C of Y, and $C \subseteq A$ for some component A of W. The maximum principle implies that $L(z) \to \infty$ on a path $\gamma \to \infty$ in C, from which it follows that F takes large values in A, both near to b, which is a pole of F, and near to infinity. This contradicts the fact that F is univalent on A, because we may analytically continue F^{-1} throughout H.

5.2. Other linear differential operators. It is reasonable to ask whether some result along the lines of the Wiman conjecture holds when the kth derivative $f^{(k)}$ is replaced by $L(f)$ as defined by (1) with real coefficients a_j.

Theorem 5.1 ([39]). *Let f be a real entire function and suppose that f and $f'' + Af$ have only real zeros, for some positive real number A. Then $f \in LP$.*

No such result holds for $A < 0$, since we may write

$$\frac{f'(z)}{f(z)} = 1 + e^{-2z}, \quad \frac{f''(z) - f(z)}{f(z)} = e^{-4z},$$

which defines an entire function f of infinite order, and so not in LP, such that f and $f'' - f$ have no zeros at all.

The proof of Theorem 5.1 needs an analogue of the Newton function used in the proof of the Wiman conjecture. For real entire f such that f and $f'' + f$ have only real zeros, write

$$L = \frac{f'}{f}, \quad T = \tan z, \quad F = \frac{TL - 1}{L + T}, \quad F' = \frac{(1 + T^2)(f'' + f)}{(L + T)^2 f}.$$

Then F' has no zeros in $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Thus F has no non-real critical values, but F might in principle have a non-real asymptotic value ($\pm i$). Also

$$F = \frac{T|L|^2 + L|T|^2 - \overline{L} - \overline{T}}{|L + T|^2}$$

so we again get Sheil-Small's property (3). Here the function F is obtained via the Wronskian method developed by Frank for Theorem 2.2 [11, 13].

6. Non-real zeros of derivatives of meromorphic functions

This section concerns the analogue for the meromorphic case of the Wiman-Pólya problem for entire functions. Let the function f be meromorphic on $\mathbb{C}$ (possibly real), and assume that (some of) the derivatives of f have only real zeros. It may then be asked whether f can be determined explicitly or, failing that, whether some bound may be given for the frequency of non-real poles, in the spirit of the Gol'dberg conjecture. A number of results were proved in the 1980s for functions with real poles, including the following.

Theorem 6.1 (Hellerstein, Rossi and Williamson [24, 52]). *Let the function f be real meromorphic on $\mathbb{C}$, not entire, with no zeros, and with only real poles. If f' and f'' have only real zeros, then $f(z) = (Az + B)^{-n}$, with $A, B \in \mathbb{R}$ and $n \in \mathbb{N}$.*

Theorem 6.2 (Hellerstein, Shen and Williamson [26]). *Let the function f be real meromorphic on $\mathbb{C}$ with only real zeros and poles, and at least one of each. Assume that f' has no zeros and f'' has only real zeros. Then f is one of*

$$A\tan(az+b)+B, \quad \frac{az+b}{cz+d}, \quad A\cdot\frac{(az+b)^2-1}{(az+b)^2},$$

where $A, B, a, b, c, d \in \mathbb{R}$.

Note that $f(z)=\tan z$ satisfies the hypotheses of Theorem 6.2. Hellerstein, Shen and Williamson [25] classified all *strictly non-real* meromorphic functions f in the plane with only real poles such that f, f' and f'' all have only real zeros. Here strictly non-real means that f is not a constant multiple of a real function.

Without the assumption of real poles, less is known. Hinkkanen considered meromorphic functions f on $\mathbb{C}$ for which *all* derivatives $f^{(k)}$ ($k \geq 0$) have only real zeros (in analogy with Pólya's conjecture for entire functions). Such functions f have at most two distinct poles, by the Pólya shire theorem, but Hinkkanen [27, 28, 29] showed that in fact they have no poles at all, apart from certain rational functions f. The following result was proved in [41].

Theorem 6.3 ([41]). *Let f be a real meromorphic function in the plane, not of the form $f=Se^P$ with S a rational function and P a polynomial. Let $k \geq 2$ be an integer. Assume that:*
(a) all but finitely many zeros of f and $f^{(k)}$ are real;
(b) the first derivative f' has finitely many zeros;
(c) there exists $M \in (0,\infty)$ such that if ζ is a pole of f of multiplicity m_ζ then

$$m_\zeta \leq M + |\zeta|^M; \tag{4}$$

(d) if $k=2$ then f'/f has finite order.

Then f satisfies

$$f(z)=\frac{R(z)e^{icz}-1}{AR(z)e^{icz}-\overline{A}}, \quad \text{where } c\in(0,\infty),\ A\in\mathbb{C}\setminus\mathbb{R}, \tag{5}$$

and

$$R \text{ is a rational function with } |R(x)|=1 \text{ for all } x\in\mathbb{R}. \tag{6}$$

Moreover, we have $k=2$ and all but finitely many poles of f are real.

Conversely, if f is given by (5) and (6) then f satisfies (a) and (b) with $k=2$.

Of course if the function f is given by $f=Se^P$ with S a rational function and P a polynomial then obviously f and all its derivatives have finitely many zeros. Theorem 6.3 is related to Theorem 6.2, but does not assume that the poles of f are real, and allows any $k \geq 2$, although $k=2$ turns out to be exceptional, and the reality of the poles then arises as a conclusion (see [47] for a similar instance of a former hypothesis becoming a conclusion). It turns out that hypotheses (c) and (d) of Theorem 6.3 may be deleted altogether, and the first derivative f' may be replaced by a higher derivative in (b).

Theorem 6.4. *Let f be a real meromorphic function in the plane, not of the form $f = Se^P$ with S a rational function and P a polynomial. Let μ and k be integers with $1 \leq \mu < k$. Assume that all but finitely many zeros of f and $f^{(k)}$ are real, and that $f^{(\mu)}$ has finitely many zeros. Then we have $\mu = 1$ and $k = 2$ and f satisfies (5) and (6). Moreover, all but finitely many poles of f are real.*

It is interesting that $k = 2$ plays an exceptional role in Theorems 6.3 and 6.4. Together with the fact that the cases $n \geq m + 3$ and $n = m + 2$ in Theorem 2.2 required different techniques in [11, 13, 14, 35], and some observations in [36, p.81], this suggests that, at least for functions with poles, the second derivative has a somewhat different character to its successors. The special nature of the first derivative was already noted in the introduction.

Theorem 6.4 will be proved in Sections 8, 9 and 10.

7. Proof of Theorem 6.4: preliminaries

The proof of Theorem 6.4 depends on some standard facts from the Wiman-Valiron theory [21]. Let the function g be transcendental and meromorphic with finitely many poles in the plane. Then there exists a rational function R with $R(\infty) = 0$ such that $h = g - R$ is a transcendental entire function. Let $h(z) = \sum_{q=0}^{\infty} \lambda_q z^q$ be the Maclaurin series of h. For $r > 0$ define

$$\mu_h(r) = \max\{|\lambda_q| r^q : q = 0, 1, 2, \ldots\}, \quad \nu(r) = \max\{q : |\lambda_q| r^q = \mu_h(r)\},$$

to be respectively the maximum term and central index of h. Let $\gamma > 1/2$. By [21, Theorems 10 and 12], there exists a set E_0 of finite logarithmic measure with the following property. Let r be large, not in E_0, and let z_0 be such that $|z_0| = r$ and $|g(z_0)| = M(r, g) = \max\{|g(z)| : |z| = r\}$. Then $|h(z_0)| \sim M(r, h)$ and we have

$$h(z) \sim \left(\frac{z}{z_0}\right)^{\nu(r)} h(z_0) \quad \text{and} \quad g(z) = h(z) + o(1) \sim h(z) \sim \left(\frac{z}{z_0}\right)^{\nu(r)} g(z_0)$$

and

$$\frac{g'(z)}{g(z)} = \frac{h'(z) + o(1)}{h(z) + o(1)} = \frac{h'(z)}{h(z)}(1 + o(1)) + \frac{o(1)}{z} \sim \frac{\nu(r)}{z}$$

for $|\log(z/z_0)| \leq \nu(r)^{-\gamma}$. We may then refer to $\nu(r)$ as the central index of g.

We require the following theorem [6] (see also [53]).

Theorem 7.1 ([6, 53])**.** *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions meromorphic on a plane domain D such that $ff^{(k)}$ has no zeros in D, for each $f \in \mathcal{F}$. Then the family $\{f'/f : f \in \mathcal{F}\}$ is normal on D.*

Next, let the function g be meromorphic in a domain containing the closed upper half-plane $\overline{H} = \{z \in \mathbb{C} : \operatorname{Im} z \geq 0\}$. For $t \geq 1$ let $\mathfrak{n}(t, g)$ be the number of poles of g, counting multiplicity, in $\{z \in \mathbb{C} : |z - it/2| \leq t/2, |z| \geq 1\}$. The Tsuji characteristic $\mathfrak{T}(r, g)$ [17, 43, 57] is defined for $r \geq 1$ by $\mathfrak{T}(r, g) = \mathfrak{m}(r, g) + \mathfrak{N}(r, g)$, where

$$\mathfrak{m}(r, g) = \frac{1}{2\pi} \int_{\sin^{-1}(1/r)}^{\pi - \sin^{-1}(1/r)} \frac{\log^+ |g(r \sin\theta e^{i\theta})|}{r \sin^2\theta} d\theta \quad \text{and} \quad \mathfrak{N}(r, g) = \int_1^r \frac{\mathfrak{n}(t, g)}{t^2} dt.$$

Lemma 7.1 ([43]). *Let the function g be meromorphic in $\overline{H}$ and assume that $\mathfrak{m}(r, g) = \mathcal{O}(\log r)$ as $r \to \infty$. Then, as $R \to \infty$,*

$$\int_R^\infty \int_0^\pi \frac{\log^+ |g(re^{i\theta})|}{r^3} \, d\theta \, dr = O\left(\frac{\log R}{R}\right).$$

8. Proof of Theorem 6.4: the growth of the logarithmic derivative

Let the integers μ and k and the function f be as in the statement of Theorem 6.4. Since f is not of the form $f = Se^P$ with S a rational function and P a polynomial, the logarithmic derivative $L = f'/f$ is transcendental. Set

$$K = \left(\frac{f}{f^{(\mu)}}\right)^k \left(\frac{f^{(k)}}{f}\right)^\mu = \frac{f^{k-\mu}(f^{(k)})^\mu}{(f^{(\mu)})^k}. \tag{7}$$

Here we observe that if K is a rational function then f and $f^{(k)}$ have finitely many zeros, since $f^{(\mu)}$ has finitely many zeros, and this implies using Theorem 2.2 that $f = Se^P$ with S a rational function and P a polynomial, contrary to hypothesis. We may therefore assume henceforth that K is transcendental. Moreover, for $j \in \mathbb{N}$, a pole of f is a pole of $f^{(j)}/f$ of multiplicity j. Thus K has finitely many poles and non-real zeros, and the logarithmic derivative K'/K is a real meromorphic function with simple poles, all but finitely many of which are real and have positive residue. Thus K'/K has a Levin-Ostrovskii factorisation

$$\frac{K'}{K} = \phi\psi. \tag{8}$$

which is determined as in [5, pp.978-979] so that: every pole of ψ is real and simple and is a pole of K'/K; the function ϕ has finitely many poles; either $\psi \equiv 1$ or ψ maps the upper half-plane $H = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ into itself. In particular ψ satisfies the Carathéodory inequality [42, Ch. I.6, Thm 8′]

$$\frac{1}{5}|\psi(i)|\frac{\sin\theta}{r} < |\psi(re^{i\theta})| < 5|\psi(i)|\frac{r}{\sin\theta} \quad \text{for} \quad r \geq 1,\, \theta \in (0, \pi). \tag{9}$$

The first task is to show that L has finite order, and the methods required will depend on whether or not k is at least 3.

Proposition 8.1. *If $k \geq 3$ then the function $L = f'/f$ has order at most 1.*

To prove Proposition 8.1, assume for the rest of this section that $k \geq 3$. Then as in [41] the fact that f and $f^{(k)}$ have finitely many non-real zeros leads to

$$\mathfrak{T}(r, L) = \mathcal{O}(\log r) \quad \text{as} \quad r \to \infty, \tag{10}$$

where $\mathfrak{T}(r, L)$ denotes the Tsuji characteristic. This is proved by means of the Wronskian method of Frank [11, 14] (see also [6, 13, 15]), but using Tsuji functionals in place of those of Nevanlinna. We write, for $j \geq 0$,

$$L_j = \frac{f^{(j+1)}}{f^{(j)}}, \quad L_{j+1} = L_j + \frac{L_j'}{L_j}. \tag{11}$$

Now let g be either K or $1/L_{\mu-1} = f^{(\mu-1)}/f^{(\mu)}$. Then (7), (10) and (11) lead at once to

$$\mathfrak{m}(r, g) = \mathcal{O}(\log r) \quad \text{as} \quad r \to \infty.$$

Hence Lemma 7.1 and the fact that g is real give

$$\int_R^\infty \frac{m(r, g)}{r^3} dr = O\left(\frac{\log R}{R}\right) \quad \text{as} \quad R \to \infty.$$

But g has finitely many poles and so we have, as $R \to \infty$,

$$\frac{T(R, g)}{R^2} \leq 2\int_R^\infty \frac{T(r, g)}{r^3} dr \leq 2\int_R^\infty \frac{m(r, g)}{r^3} dr + O\left(\frac{\log R}{R^2}\right) = O\left(\frac{\log R}{R}\right).$$

Thus g has order at most 1. It now follows using (11) that L_j has order at most 1 for all $j \geq \mu - 1$. But (7) gives the formula

$$\frac{K'}{K} = (k - \mu)\frac{f'}{f} + \mu\frac{f^{(k+1)}}{f^{(k)}} - k\frac{f^{(\mu+1)}}{f^{(\mu)}},$$

which now shows that $L = f'/f$ has order at most 1. This completes the proof of Proposition 8.1.

9. The growth of the logarithmic derivative when $k = 2$

Assume throughout this section that $k = 2$. In this case Frank's method as employed in the proof of Proposition 8.1 is not available, and an alternative approach based on Wiman-Valiron theory and normal families will be used.

Proposition 9.1. *The function $L = f'/f$ has finite order.*

To prove Proposition 9.1, observe first that since $k = 2$ we must have $\mu = 1$. Write

$$F(z) = z - \frac{f(z)}{f'(z)} = z - \frac{1}{L(z)}, \quad F' = \frac{ff''}{(f')^2} = K, \tag{12}$$

where K is defined by (7) with $\mu = 1$, $k = 2$.

Lemma 9.1. *Suppose that ϕ is a rational function in (8). Then K has finite order and so have F and L.*

Proof. The following argument is from [5, pp.989-990]. The function K is transcendental with finitely many poles: as in Section 7 let $\nu(r)$ denote the central index of K. Then there exist a set E_0 of finite logarithmic measure and, for each $r \in [1, \infty) \setminus E_0$, a point z_0 such that $|z_0| = r$ and

$$\frac{K'(z)}{K(z)} \sim \frac{\nu(r)}{z} \quad \text{for} \quad z = z_0 e^{it}, \ -\nu(r)^{-2/3} \leq t \leq \nu(r)^{-2/3}.$$

This leads at once to

$$\int_0^{2\pi} \left| \frac{K'(re^{it})}{K(re^{it})} \right|^{5/6} dt \geq \nu(r)^{1/6} r^{-5/6} \quad \text{as} \quad r \to \infty \quad \text{with} \quad r \notin E_0.$$

But (8) and (9) give, for some positive absolute constant c,

$$\int_0^{2\pi} \left| \frac{K'(re^{it})}{K(re^{it})} \right|^{5/6} dt \leq cM(r, \phi)^{5/6} |\psi(i)|^{5/6} r^{5/6} \quad \text{as} \quad r \to \infty.$$

It follows that $\nu(r) \leq M(r, \phi)^5 r^{11}$ as $r \to \infty$ with $r \notin E_0$. Hence K has finite order, and so have F and L by (12). ■

Assume for the remainder of this section that ϕ is transcendental in (8). It will be shown that this assumption leads to a contradiction, which in view of Lemma 9.1 will establish Proposition 9.1. Denote by $N = N(r)$ the central index of ϕ. Then there exists a set E_1 of finite logarithmic measure such that, if $r \in [1, \infty) \setminus E_1$ and $|z_1| = r$, $|\phi(z_1)| = M(r, \phi)$ then

$$\phi(z) \sim \phi(z_1) \left(\frac{z}{z_1} \right)^{N(r)} \quad \text{for} \quad \left| \log \frac{z}{z_1} \right| \leq N(r)^{-7/12}.$$

Since ϕ is real it follows that for large $r \in [1, \infty) \setminus E_1$ there exists z_0 with

$$|z_0| = r, \quad N(r)^{-2/3} \leq \theta_0 = \arg z_0 \leq \pi - N(r)^{-2/3}, \quad |\phi(z_0)| \sim M(r, \phi), \tag{13}$$

such that

$$\phi(z) \sim \phi(z_0) \left(\frac{z}{z_0} \right)^{N(r)} \quad \text{for} \quad \left| \log \frac{z}{z_0} \right| \leq N(r)^{-3/4}. \tag{14}$$

Denote by c_j, d_j positive constants which are independent of r.

Lemma 9.2. *There exists $c_1 \geq 1$ with the following property. Let R_0 be large and positive and for large $r \in [1, \infty) \setminus E_1$ let Ω_r be the shorter arc of the circle $|z| = r$ joining z_0 to ir, and let*

$$\Sigma_r = \Omega_r \cup \{it : R_0 \leq t \leq r\}.$$

Then

$$\max\{|K'(z)/K(z)| : z \in \Sigma_r\} \leq q(r) = M(r, \phi) N(r)^{c_1} |\psi(z_0)| \to \infty \tag{15}$$

as $r \to \infty$ with $r \in [1, \infty) \setminus E_1$.

Proof. We may assume that $\psi(H) \subseteq H$, since otherwise (15) follows immediately from (8) and the fact that ϕ has finitely many poles. Let $r \in [1, \infty) \setminus E_1$ be large. For convenience we assume that $N(r)^{-2/3} \leq \theta_0 = \arg z_0 \leq \pi/2$ in (13); the proof in the contrary case requires only trivial modifications. Then (9) and (13) give

$$|\psi(z_0)| \geq \frac{|\psi(i)| \sin\left(N(r)^{-2/3}\right)}{5r} \geq \frac{c_2}{rN(r)^{2/3}}, \tag{16}$$

so that $q(r) \to \infty$ in (15), since $c_1 \geq 1$ and ϕ is transcendental.

We may write $\log \psi(z)$ as a function of $\zeta = \log z$ on H, which then maps the strip $0 < \operatorname{Im} \zeta < \pi$ into itself. Bloch's theorem gives

$$\left| \frac{d \log \psi(re^{i\theta})}{d\theta} \right| \leq \frac{c_3}{\theta} \quad \text{and} \quad \left| \log \frac{\psi(re^{i\theta})}{\psi(z_0)} \right| \leq c_3 \log \frac{\theta}{\theta_0} \leq c_4 \log N(r) \tag{17}$$

for $\theta_0 \leq \theta \leq \pi/2$, where $c_4 \geq 1$. In view of (8) it then follows that

$$\max\{|K'(z)/K(z)| : z \in \Omega_r\} \leq q(r).$$

Now let $1 \leq t \leq r$. Applying Bloch's theorem again gives

$$\left| \frac{d \log \psi(is)}{d \log s} \right| \leq c_5$$

for $1 \leq s \leq r$ and so, using (17),

$$\left| \log \frac{\psi(it)}{\psi(z_0)} \right| \leq \left| \log \frac{\psi(it)}{\psi(ir)} \right| + \left| \log \frac{\psi(ir)}{\psi(z_0)} \right| \leq c_5 \log \frac{r}{t} + c_4 \log N(r). \tag{18}$$

Let $Q > c_5$ be an integer. Then there exists a rational function S_0, with a pole of multiplicity at most $Q - 1$ at infinity, such that

$$\phi_2 = \phi - S_0 \quad \text{and} \quad \phi_1(z) = \frac{\phi_2(z)}{z^Q}$$

are transcendental entire functions. Let R_0 be large and positive. Then for $R_0 \leq t \leq r$ we have

$$\begin{aligned} M(t, \phi) &\leq t^Q M(t, \phi_1) + M(t, S_0) \leq 2t^Q M(r, \phi_1) \\ &\leq 2 \left(\frac{t}{r}\right)^Q M(r, \phi_2) \leq 4 \left(\frac{t}{r}\right)^Q M(r, \phi). \end{aligned}$$

Combining this with (8) and (18) then yields, since r is large and $Q > c_5$,

$$\left| \frac{K'(it)}{K(it)} \right| \leq 4 \left(\frac{t}{r}\right)^Q M(r, \phi) \left(\frac{r}{t}\right)^{c_5} N(r)^{c_4} |\psi(z_0)| \leq q(r),$$

which completes the proof of (15).

∎

The estimate (15) and integration of K'/K now lead to

(19) $$|\log|K(z)|| \le c_6 rq(r) + |\log|K(iR_0)|| \le 2c_6 rq(r) \quad \text{for all } z \in \Sigma_r.$$

Lemma 9.3. *Let $r \in [1,\infty) \setminus E_1$ be large. Then*

(20) $$\log|K(z)| \le c_7 rq(r) \quad \textit{for} \quad |z - z_0| \le \frac{r}{N(r)^{3/4}}.$$

Proof. Set

(21) $$R_1 = \frac{4r}{N(r)^{3/4}}.$$

The relations (12) and integration of (19) yield

(22) $$\log\left|\frac{f(z_0)}{f'(z_0)}\right| \le c_8 rq(r).$$

Let

$$g(z) = f(z_0 + R_1 z), \quad G(z) = \frac{g(z)}{g'(z)} = \frac{f(z_0 + R_1 z)}{R_1 f'(z_0 + R_1 z)}$$

for $|z| < 1$. Then (15), (21) and (22) imply that

(23) $$\log|G(0)| \le \log\left|\frac{f(z_0)}{f'(z_0)}\right| + \log\frac{1}{R_1} \le d_1 rq(r).$$

By (13) and (21) the functions g and g'' have no zeros in $|z| < 1$, and so Theorem 7.1 gives $d_2 > 0$, independent of r, such that

$$\frac{|G'(z)|}{1 + |G(z)|^2} \le d_2 \quad \text{for} \quad |z| \le \frac{3}{4}.$$

Hence (23) and [20, p.12] lead to

$$T(3/4, G) \le T_0(3/4, G) + d_3 + \log^+|G(0)| \le d_4 + \log^+|G(0)| \le 2d_1 rq(r),$$

where T_0 denotes the Ahlfors characteristic. Since G is analytic on $|z| < 1$ by (13), it follows that

$$\log|G(z)| \le d_5 rq(r) \quad \text{for} \quad |z| \le \frac{1}{2}.$$

This implies in turn that

$$\log\left|\frac{f(z)}{f'(z)}\right| \le d_5 rq(r) + \log R_1 \le d_5 rq(r) + \log r \le 2d_5 rq(r)$$

for $|z - z_0| \le R_1/2$. Thus (12) and Cauchy's estimate for derivatives yield

$$\log|F(z)| \le 3d_5 rq(r), \quad \log|K(z)| \le 3d_5 rq(r) + d_6 + \log\frac{1}{R_1} \le 4d_5 rq(r)$$

for $|z - z_0| \le R_1/4$. In view of (21), this proves (20).

∎

To obtain a contradiction let $r \in [1, \infty) \setminus E_1$ be large, and recall the estimate (14) for $\phi(z)$ near z_0. For z as in (14), Bloch's theorem, (8) and (13) yield

$$\frac{N(r)^{-2/3}}{2} \leq \arg z \leq \pi - \frac{N(r)^{-2/3}}{2}, \quad \left| \frac{d \log \psi(z)}{d \log z} \right| \leq d_8 N(r)^{2/3},$$

and

$$\psi(z) \sim \psi(z_0), \quad \frac{K'(z)}{K(z)} = \phi(z_0)\psi(z_0) \left(\frac{z}{z_0} \right)^{N(r)} (1 + \varepsilon(z)), \quad \varepsilon(z) = o(1). \tag{24}$$

Choose z with

$$\left| \arg \frac{z}{z_0} \right| \leq N(r)^{-11/12}, \quad \log \left| \frac{z}{z_0} \right| = N(r)^{-5/6}, \tag{25}$$

and denote the straight line segment from z_0 to z by L_z. Then simple geometric considerations show that the angle between L_z and the ray $\arg t = \arg z_0$ is small. Thus integration along L_z gives, with $N = N(r)$ and $s = |t|$,

$$\begin{aligned} \int_{L_z} t^N (1 + \varepsilon(t))\, dt &= \frac{1}{N+1}(z^{N+1} - z_0^{N+1}) + o\left(\int_{L_z} |t|^N \, |dt| \right) \\ &= \frac{1}{N+1}(z^{N+1} - z_0^{N+1}) + o\left(\int_{|z_0|}^{|z|} s^N \, ds \right) \sim \frac{z^{N+1}}{N+1}. \end{aligned}$$

Hence (13) and (24) yield, for z as in (25),

$$w = \log \frac{K(z)}{K(z_0)} \sim \frac{z_0 \phi(z_0)\psi(z_0)}{N+1} \left(\frac{z}{z_0} \right)^N, \quad |w| \geq \frac{rM(r, \phi)|\psi(z_0)|e^{N^{1/6}}}{2N}.$$

Moreover the variation of $\arg w$ as z varies over the arc given by (25) exceeds 2π, and so z may be chosen such that w is real and positive. But then (19) and (20) give

$$\begin{aligned} c_7 r q(r) \geq \log |K(z)| &\geq \log |K(z_0)| + \frac{rM(r, \phi)|\psi(z_0)|e^{N^{1/6}}}{2N} \\ &\geq -2c_6 r q(r) + \frac{rM(r, \phi)|\psi(z_0)|e^{N^{1/6}}}{2N}. \end{aligned}$$

Since $q(r)$ is given by (15) and $N = N(r)$ tends to infinity, this is a contradiction, and Proposition 9.1 is proved.

10. Proof of Theorem 6.4: the growth of f

Proposition 10.1. *The function f has finite order.*

Proof. Assume that f has infinite order. Propositions 8.1 and 9.1 show that $L = f'/f$ has finite order and so has the transcendental function K defined by

(7). Moreover, the zeros of f have finite exponent of convergence, since $f^{(\mu)}$ has finitely many zeros. Hence there exist entire functions f_1 and f_2 such that

$$f = \frac{f_1}{f_2}, \quad L = \frac{f'}{f} = \frac{f_1'}{f_1} - \frac{f_2'}{f_2}, \qquad \rho(f_1) < \infty = \rho(f_2). \tag{26}$$

By standard estimates for logarithmic derivatives [18] there exist $M_1 > 0$ and a set $E_2 \subseteq [1, \infty)$ of finite logarithmic measure such that

$$\left| \frac{L^{(j)}(z)}{L(z)} \right| + \left| \frac{K'(z)}{K(z)} \right| + \left| \frac{K'(z)}{K(z) - 1} \right| + \left| \frac{f_1'(z)}{f_1(z)} \right| \leq |z|^{M_1} \tag{27}$$

for $j = 1, \ldots, k$ and $|z| = r \in [1, \infty) \setminus E_2$.

Let $\nu(r)$ denote the central index of f_2. Since $\rho(f_2) = \infty$ and $\nu(r)$ is increasing, the Wiman-Valiron theory (see Section 7) gives an unbounded set $F_0 \subseteq [1, \infty) \setminus E_2$ with the following properties. We have

$$\lim_{r \to \infty, r \in F_0} \frac{\log \nu(r)}{\log r} = \infty. \tag{28}$$

Moreover, for each $r \in F_0$ there exists z_0 such that $|z_0| = r$ and

$$\frac{f_2'(z)}{f_2(z)} \sim \frac{\nu(r)}{z} \quad \text{for} \quad z = z_0 e^{it}, \ -\nu(r)^{-2/3} \leq t \leq \nu(r)^{-2/3}.$$

In view of (27) and (28) we obtain an interval ω_r, of length $2\nu(r)^{-2/3}$, such that

$$L(z) = \frac{f'(z)}{f(z)} \sim \frac{\nu(r)}{z} \quad \text{for} \quad z = re^{it}, \ t \in \omega_r, \ r \in F_0. \tag{29}$$

This leads at once to

$$\int_{\omega_r} \left| L(re^{it}) \right|^{5/6} dt \geq \nu(r)^{1/6} r^{-5/6} \quad \text{as} \quad r \to \infty \quad \text{with} \quad r \in F_0. \tag{30}$$

Let c denote a positive constant, not necessarily the same at each occurrence, but independent of r, and let $j \in \mathbb{N}$. By (27), (28), (29) and the well known representation [20, Lemma 3.5]

$$\frac{f^{(j)}}{f} = L^j + P_{j-1}[L], \tag{31}$$

where $P_{j-1}[L]$ is a polynomial in L and its derivatives, of degree at most $j - 1$, we have

$$\begin{aligned} \frac{f^{(j)}(z)}{f(z)} &= L(z)^j \left(1 + \frac{P_{j-1}[L](z)}{L(z)^j} \right) \\ &= L(z)^j \left(1 + \frac{\mathcal{O}(r^c)}{L(z)} \right) \quad \text{for} \quad z = re^{it}, \ t \in \omega_r, \ r \in F_0. \end{aligned}$$

Hence using (7) we deduce that

$$K(z)-1=\frac{\mathcal{O}(r^c)}{L(z)} \quad \text{for} \quad z=re^{it},\, t\in\omega_r,\, r\in F_0.$$

Writing

$$\frac{1}{K-1}=\frac{K}{K'}\left(\frac{K'}{K(K-1)}\right)=\frac{K}{K'}\left(\frac{K'}{K-1}-\frac{K'}{K}\right)$$

we now obtain, using (8) and (27) again,

$$L(z)=\frac{\mathcal{O}(r^c)}{K(z)-1}=\frac{\mathcal{O}(r^c)K(z)}{K'(z)}=\frac{\mathcal{O}(r^c)}{\phi(z)\psi(z)} \quad \text{for} \quad z=re^{it},\, t\in\omega_r,\, r\in F_0.$$

But K is transcendental of finite order and so (8), (9) and the inequality

$$m(r,\phi)\le m(r,K'/K)+m(r,1/\psi)$$

imply that ϕ is a rational function, not identically zero. Hence (9) gives

$$\int_{\omega_r}\left|L(re^{it})\right|^{5/6}dt=\mathcal{O}(r^c) \quad \text{for} \quad r\in F_0,$$

which contradicts (28) and (30). This proves Proposition 10.1.

■

We may now finish the proof of Theorem 6.4. Suppose first that $\mu=1$. Since f has finite order and

$$n(r,f)=\mathcal{O}(N(2r,f))=\mathcal{O}(T(2r,f))$$

as $r\to\infty$, the hypotheses of Theorem 6.3 are satisfied, and the result follows at once.

Assume henceforth that $\mu\ge 2$, so that $k\ge 3$. This time the fact that f has finite order implies that f has finitely many poles, using Theorem 2.3. Hence $f^{(\mu)}=S_\mu e^{Q_\mu}$, with S_μ a rational function and Q_μ a non-constant real polynomial.

Suppose first that Q_μ has degree 1. In this case integrating μ times shows that $f(z)=T_1(z)+T_2(z)e^{a_1 z}$ with T_1 a polynomial, $T_2\not\equiv 0$ a rational function, and $a_1\in\mathbb{R}\setminus\{0\}$. Since L is transcendental we must have $T_1\not\equiv 0$. But elementary considerations now show that f has infinitely many non-real zeros, which is a contradiction.

We may therefore assume that Q_μ has degree $q_\mu\ge 2$. Since L and $f/f^{(\mu)}$ have order at most 1, by Proposition 8.1, it follows that we may write $f=\Pi e^{Q_\mu}$, where Π is meromorphic with finitely many poles and with order at most 1. Let $\varepsilon\in(0,1)$ be small. Then Gundersen's estimates for logarithmic derivatives [18] give rise to a set $E_3\subseteq[1,\infty)$ of finite logarithmic measure such that

$$\left|\frac{L^{(j)}(z)}{L(z)}\right|+\left|\frac{\Pi'(z)}{\Pi(z)}\right|\le |z|^{\varepsilon} \quad \text{and} \quad L(z)=\frac{\Pi'(z)}{\Pi(z)}+Q_\mu'(z)\sim Q_\mu'(z)$$

for $j = 1, \dots, k$ and $|z| = r \in [1, \infty) \setminus E_3$. But for these z we then have, on recalling (31),

$$\frac{f^{(\mu)}(z)}{f(z)} = L(z)^\mu \left(1 + \frac{P_{\mu-1}[L](z)}{L(z)^\mu} \right) \sim L(z)^\mu \sim Q'_\mu(z)^\mu.$$

Hence $f^{(\mu)}(z)/f(z)$, which has finitely many zeros, must be a rational function, and consequently so must Π, which contradicts the assumption that f is not of the form Se^P with S a rational function and P a polynomial. The case $\mu \geq 2$ is therefore impossible, which completes the proof of Theorem 6.4.

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