

# Derivatives of meromorphic functions of finite order

J.K. Langley

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**Abstract** A result is proved concerning meromorphic functions  $f$  of finite order in the plane such that all but finitely many zeros of  $f''$  are zeros of  $f'$ .

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## 1 Introduction

The starting point of this paper is the following theorem from [9].

**Theorem 1** ([9]) *Assume that the function  $f$  is meromorphic of finite lower order in the plane and that  $f^{(k)}$  has finitely many zeros, for some  $k \geq 2$ . Assume further that there exists a positive real number  $M$  such that if  $\zeta$  is a pole of  $f$  of multiplicity  $m_\zeta$  then*

$$m_\zeta \leq M + |\zeta|^M. \quad (1)$$

*Then  $f$  has finitely many poles.*

Condition (1) is evidently satisfied if  $f$  has finite order (see [4, p.16] for the definitions of order and lower order). Theorem 1 fails for  $k = 1$ , as shown by simple examples, and for  $k \geq 2$  and infinite lower order, in which case an example is constructed in [8] with infinitely many poles, all simple, such that  $f^{(k)}$  has no zeros at all. The result was inspired by the conjecture made by A.A. Gol'dberg, to the effect that for  $k \geq 2$  and a meromorphic function  $f$  in the plane, regardless of growth, the frequency of distinct poles of  $f$  is controlled

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J.K. Langley  
School of Mathematical Sciences, University of Nottingham, NG7 2RD, UK.  
Tel.: +44-115-9514964  
Fax: +44-115-9514951  
E-mail: jkl@maths.nott.ac.uk

by the frequency of zeros of  $f^{(k)}$ , up to an error term which is small compared to the Nevanlinna characteristic. Yamanoi has now proved this conjecture in a landmark paper [14]; however, because of the error terms involved, his result does not imply Theorem 1 directly.

This paper is concerned with a generalisation of Theorem 1 in a different direction. The assumption there that  $f^{(k)}$  has finitely many zeros is a strong one, so that it is natural to ask whether it may be replaced by something less restrictive. A reasonable candidate is the condition that all but finitely many zeros of  $f^{(k)}$  have the same image under  $f^{(k-1)}$ , which may then be assumed to be 0, but the following example shows that this does not by itself imply that  $f$  has finitely many poles. Set

$$f(z) = z - \tan z, \quad f'(z) = -\tan^2 z, \quad f''(z) = -2 \tan z \sec^2 z. \quad (2)$$

Here all zeros of  $f''$  are zeros of  $f'$  and fixpoints of  $f$ , all zeros and poles of  $f'$  have the same multiplicity, and 1 is an asymptotic value of  $f'$ . More generally it may be observed that, for any even positive integer  $n$ , the antiderivative of  $\tan^n z$  is meromorphic in  $\mathbb{C}$ . The example (2) shows that the following theorem, which evidently implies Theorem 1, is essentially sharp.

**Theorem 2** *Let  $k \geq 2$  be an integer and let  $f$  be a meromorphic function of finite lower order in the plane with the following properties:*

- (i) *the zeros of  $f^{(k-1)}$  have bounded multiplicities;*
- (ii) *all but finitely many zeros of  $f^{(k)}$  are zeros of  $f^{(k-1)}$ ;*
- (iii) *there exists  $M \in (0, +\infty)$  such that if  $\zeta$  is a pole of  $f$  of multiplicity  $m_\zeta$  then (1) holds;*
- (iv) *for each  $\varepsilon > 0$ , all but finitely many zeros  $z$  of  $f^{(k)}$  are such that either  $|f^{(k-2)}(z)| \leq \varepsilon|z|$  or  $\varepsilon|f^{(k-2)}(z)| \geq |z|$ .*

*Then  $f^{(k)}$  has a representation  $f^{(k)} = Re^P$  with  $R$  a rational function and  $P$  a polynomial. In particular,  $f$  has finite order and finitely many poles, and  $f^{(k)}$  has finitely many zeros.*

It suffices to prove Theorem 2 for  $k = 2$  and, as already noted, condition (iii) holds when  $f$  has finite order. If  $f$  is a meromorphic function of finite lower order in the plane satisfying condition (ii) of Theorem 2, with  $k = 2$ , then  $f'$  has finitely many critical values and so finitely many asymptotic values, by a result of Bergweiler and Eremenko [2] and its extension by Hinchliffe [7] to functions of finite lower order (see Section 3). Therefore Theorem 2 follows from the next result, which fails for infinite lower order, because of the same example from [8] mentioned after Theorem 1.

**Theorem 3** *Let  $f$  be a meromorphic function of finite lower order in the plane satisfying conditions (i), (ii) and (iii) of Theorem 2, with  $k = 2$ . Assume that there exist positive real numbers  $\kappa$  and  $R_0$  such that if  $z$  is a zero of  $f''$  with  $|z| \geq R_0$  then  $|f(z) - \alpha z| \geq \kappa|z|$  for all finite non-zero asymptotic values  $\alpha$  of  $f'$ . Then  $f'' = Re^P$  with  $R$  a rational function and  $P$  a polynomial.*

## 2 Lemmas needed for Theorem 3

Throughout this paper  $B(z_0, r)$  will denote the disc  $\{z \in \mathbb{C} : |z - z_0| < r\}$  and  $S(z_0, r)$  will be the circle  $\{z \in \mathbb{C} : |z - z_0| = r\}$ . The following results are both well known.

**Lemma 1** ([13], p.116) *Let  $D$  be a simply connected domain in  $\mathbb{C}$  not containing the origin, and let  $z_0$  lie in  $D$ . Let  $r$  satisfy  $0 < 4r < |z_0|$  or  $4|z_0| < r < \infty$ . Let  $\theta(t)$  denote the angular measure of  $D \cap S(0, t)$ , and let  $D_r$  be the component of  $D \setminus S(0, r)$  which contains  $z_0$ . Then the harmonic measure of  $S(0, r)$  with respect to the domain  $D_r$ , evaluated at  $z_0$ , satisfies*

$$\omega(z_0, S(0, r), D_r) \leq C \exp\left(-\pi \int_I \frac{dt}{t\theta(t)}\right), \quad (3)$$

with  $C$  an absolute constant,  $I = [2|z_0|, r/2]$  if  $4|z_0| < r$ , and  $I = [2r, |z_0|/2]$  if  $4r < |z_0|$ .

**Lemma 2** ([6], p.366) *Let  $Q$  be a positive integer and let  $w_1, \dots, w_Q$  be complex numbers. For each  $A > 0$  the estimate*

$$\prod_{j=1}^Q |z - w_j| \geq A^Q \quad (4)$$

holds for all  $z$  outside a union of discs having sum of radii at most  $6A$ .

## 3 Critical points and asymptotic values

Suppose that the function  $h$  is transcendental and meromorphic in the plane, and that  $h(z)$  tends to  $a \in \mathbb{C}$  as  $z$  tends to infinity along a path  $\gamma$ . Then  $a$  is an asymptotic value of  $h$ , and the inverse function  $h^{-1}$  has a transcendental singularity over  $a$  [2, 11]. For each  $t > 0$ , let  $C(t)$  be that component of the set  $\{z \in \mathbb{C} : |h(z) - a| < t\}$  which contains an unbounded subpath of  $\gamma$ . The singularity of  $h^{-1}$  over  $a$  corresponding to  $\gamma$  is called direct [2] if  $C(t)$  contains no zeros of  $h(z) - a$ , for some  $t > 0$ . Singularities over  $\infty$  are classified analogously.

Recall next some standard facts from [11, p.287]. Let  $G$  be a transcendental meromorphic function in the plane with no asymptotic or critical values in the annulus  $1 < |w| < \infty$ . Then every component  $C_0$  of  $\{z \in \mathbb{C} : |G(z)| > 1\}$  is simply connected, and there are two possibilities. Either (i)  $C_0$  contains one pole  $z_0$  of  $G$  of multiplicity  $k$ , in which case  $G^{-1/k}$  maps  $C_0$  univalently onto  $B(0, 1)$ , or (ii)  $C_0$  contains no pole of  $G$ , but instead a path tending to infinity on which  $G$  tends to infinity. In case (ii) the function  $w = \log G(z)$  maps  $C_0$  univalently onto the right half plane.

**Lemma 3 ([9])** *There exists a positive absolute constant  $c_0$  with the following property. Suppose that  $G$  is a transcendental meromorphic function in the plane and that  $G'$  has no asymptotic values and no critical values in the annulus  $0 < |w| < d_1 < \infty$ . Let  $D$  be a component of the set  $\{z \in \mathbb{C} : |G'(z)| < d_1\}$  on which  $G'$  has no zeros, but such that  $D$  contains a path tending to infinity on which  $G'(z)$  tends to 0. If  $z_1$  is in  $D$  and  $\log |d_1/G'(z_1)| \geq 1$  then*

$$|G(z_1)| \leq S + \frac{c_0 |z_1 G'(z_1)|}{\log |d_1/G'(z_1)|},$$

in which the positive constant  $S$  depends on  $G$  and  $D$  but not on  $z_1$ .

□

Suppose next that the function  $F$  is meromorphic of finite lower order in the plane, and that all but finitely many zeros of  $F'$  are zeros of  $F$ . Then  $F$  has finitely many critical values. By Hinchliffe's extension [7] to the finite lower order case of a theorem of Bergweiler and Eremenko [2], the function  $F$  has finitely many asymptotic values. Furthermore, all asymptotic values of  $F$  give rise to direct transcendental singularities of the inverse function  $F^{-1}$  and, by the Denjoy-Carleman-Ahlfors theorem [2, 6, 11], there are finitely many such singularities. The following facts are related to the argument from [8, Section 4]. Let  $J$  be a polygonal Jordan curve in  $\mathbb{C} \setminus \{0\}$  such that every finite non-zero critical or asymptotic value of  $F$  lies on  $J$ , but is not a vertex of  $J$ , and such that the complement of  $J$  in  $\mathbb{C} \cup \{\infty\}$  consists of two simply connected domains  $B_1$  and  $B_2$ , with  $0 \in B_1$  and  $\infty \in B_2$ . Fix conformal mappings

$$h_m : B_m \rightarrow B(0, 1), \quad m = 1, 2, \quad h_1(0) = 0, \quad h_2(\infty) = 0. \quad (5)$$

The mapping  $h_1$  may then be extended to be quasiconformal on the plane [12, Ch.5], fixing infinity, and there exist a meromorphic function  $G$  and a quasiconformal mapping  $\psi$  such that  $h_1 \circ F = G \circ \psi$  on  $\mathbb{C}$ .

**Lemma 4** *For  $j = 1, 2$ , all components of  $F^{-1}(B_j)$  are simply connected and all but finitely many are unbounded. If  $C$  is a component of  $F^{-1}(B_1)$  then exactly one of the following two conclusions holds: either  $C$  contains one zero of  $F$ , of multiplicity  $k \geq 1$ , and  $C$  is mapped  $k$  to 1 onto  $B_1$  by  $F$ ; or  $C$  contains no zeros of  $F$ , but does contain a path tending to infinity on which  $F(z)$  tends to 0, while  $w = \log h_1(F(z))$  maps  $C$  univalently onto the half plane  $\operatorname{Re} w < 0$  and  $F$  is not univalent on  $C$ .*

*Furthermore, if a zero  $z_0$  of  $F'$  lies in a component  $C$  of  $F^{-1}(B_1)$  then  $z_0$  is a zero of  $F$  and in particular  $F(C) = B_1$  and  $z_0$  is the only zero of  $F'$  in  $C$ .*

*Similarly, every component of  $F^{-1}(B_2)$  contains either one pole of  $F$ , of multiplicity  $p \geq 1$ , or a path tending to infinity on which  $F(z)$  tends to  $\infty$ .*

*Proof.* All finite non-zero singular values of  $G^{-1}$  have modulus 1, and all but finitely many zeros of  $G'$  are zeros of  $G$ , since  $h_1(0) = 0$ . Thus all these

assertions follow from the classification of components of  $\{z \in \mathbb{C} : |G(z)| > 1\}$ , as discussed prior to Lemma 3, and the matching properties of components of  $\{z \in \mathbb{C} : |G(z)| < 1\}$ .  $\square$

#### 4 Proof of Theorem 3: first part

Let the function  $f$  be as in the hypotheses. If  $f''/f'$  is a rational function then  $f'$  is a rational function multiplied by the exponential of a polynomial, and so is  $f''$ . Assume henceforth that  $f''/f'$  is transcendental: then obviously so is  $f$ . Apply the reasoning and notation of Section 3, with  $F = f'$ . The following is an immediate consequence of Lemma 3.

**Lemma 5** *Arbitrarily small positive real numbers  $\varepsilon_1$  and  $\varepsilon_2$  may be chosen with the following properties. There exist finitely many unbounded simply connected domains  $U_n$ , each a component of the set  $\{z \in \mathbb{C} : |f'(z) - b_n| < \varepsilon_1\}$ , such that  $U_n$  contains a path tending to infinity on which  $f'(z)$  tends to the finite asymptotic value  $b_n$ . Here  $f'(z) \neq b_n$  on  $U_n$  and  $|f(z) - b_n z| < \varepsilon_2 |z|$  for all large  $z$  in  $U_n$  (that is, for all  $z \in U_n$  with  $|z|$  large enough). If  $\Gamma$  is a path tending to infinity on which  $f'$  tends to a finite asymptotic value  $\alpha$ , then there exists  $n$  such that  $\alpha = b_n$  and  $\Gamma \setminus U_n$  is bounded. The  $b_n$  need not be distinct, and some of them may be 0.*

$\square$

**Lemma 6** *There exists a positive real number  $s_1 < \varepsilon_1$  with the following property. Let  $b_p$  be a finite non-zero asymptotic value of  $f'$ . Then the conformal map  $h_1 : B_1 \rightarrow B(0, 1)$  extends to be analytic and univalent on  $B_1 \cup B(b_p, s_1)$ .*

*Proof.* This follows from the reflection principle and the fact that each non-zero  $b_p$  lies on the polygonal Jordan curve  $J = \partial B_1$  but is not a vertex of  $J$ .  $\square$

**Definition 1** Fix positive real numbers  $\rho, \sigma$  and  $\tau$  with  $\tau < s_1 < \varepsilon_1$  and  $\sigma/\tau$  and  $\rho/\sigma$  small. Fix  $W_0 \in \mathbb{C}$  such that  $f'(W_0)$  is large, so that in particular  $f'(W_0) \in B_2$ .

**Lemma 7** *With the notation of Definition 1, there exist positive real numbers  $M_1, M_2, M_3$  having the following properties. Let  $z_0$  be large with  $|f'(z_0)| < \tau$  and assume that  $z_0$  lies in a component  $C$  of  $(f')^{-1}(B_1)$  satisfying one of the following two conditions:*

- (A) *there is at least one zero of  $f''$  in  $C$ ;*
- (B) *the function  $f'$  is univalent on  $C$ , and  $C \cap U_p$  and  $C \cap U_q$  are both non-empty, where  $U_p$  and  $U_q$  are as in Lemma 5 with  $0 \neq b_p \neq b_q \neq 0$ .*

*Then  $|z_0 f''(z_0)| \leq M_1$  and there exists a disc  $B(z_0^*, M_2 |z_0^*|) \subseteq B(z_0, \frac{1}{2} |z_0|) \cap C$  on which*

$$\left| \frac{f'''(\zeta)}{f''(\zeta)} \right| \leq \frac{M_3}{|\zeta|}. \quad (6)$$

*Proof.* Observe that conditions (A) and (B) are mutually exclusive. Denote positive constants by  $c_j$  and small positive constants by  $\delta_j$ ; these will be independent of  $z_0$  and  $C$ . Applying Lemma 4 with  $F = f'$  shows that in case (A) there is exactly one point in  $C$  at which  $f''$  vanishes, and it must be a zero of  $f'$ ; moreover, in both cases  $f'(C) = B_1$  and  $C$  contains precisely one zero of  $f'$ , of multiplicity  $m$ , with  $m = 1$  in Case (B). In either case,  $m \leq c_1$ , by hypothesis (i) of Theorem 2 (and hence also of Theorem 3). There exist only finitely many components  $C_1$  of  $(f')^{-1}(B_1)$  which are bounded or have a zero of  $f''$  on their boundary, and if one of these contains a zero of  $f'$  then the set  $\{z \in C_1 : |f'(z)| \leq \tau\}$  is compact. Therefore since  $z_0$  is large the component  $C$  is unbounded and simply connected and its boundary  $\partial C$  contains no zeros of  $f''$ . Now set  $v_0 = (h_1 \circ f')^{1/m}$ , with  $h_1$  as in (5). Then  $v_0$  maps  $C$  univalently onto  $B(0, 1)$ , and  $u_0 = v_0(z_0)$  satisfies  $|u_0| \leq \delta_1$ , since  $m \leq c_1$  and  $\tau$  is small.

Let  $\Gamma$  be a component of  $\partial C$ . Then  $\Gamma$  is a simple curve tending to infinity in both directions and, as  $z$  tends to infinity in either direction along  $\Gamma$ , the image  $f'(z)$  must tend to a finite non-zero asymptotic value of  $f'$ ; this is because  $v_0$  is univalent on  $C$ . Hence there exists  $z_1$  lying close to  $\Gamma$ , such that  $z_1 \in C \cap U_n$ , for some  $U_n$  as in Lemma 5, with  $b_n \neq 0$  and  $|f'(z_1) - b_n| < \varepsilon_1$ . By construction,  $b_n$  lies on the polygonal Jordan curve  $J$  but is not a vertex of  $J$ . Thus analytic continuation of  $(f')^{-1}$  along a path in the semi-disc  $B(b_n, \varepsilon_1) \cap B_1$  then gives a point  $z_2 \in C \cap U_n$  with  $|f'(z_2) - b_n| < \varepsilon_1$ , as well as  $|h_1(f'(z_2))| \leq 1 - \delta_2$ , which implies in turn that  $|v_0(z_2)| \leq 1 - \delta_3$ .

Let  $G_0 : B(0, 1) \rightarrow C$  be the inverse function of  $v_0$ , and suppose that  $u_0$  satisfies  $G'_0(u_0) = o(|z_0|)$ . Then Koebe's distortion theorem implies that  $G'_0(u) = o(|z_0|)$  for  $|u| \leq 1 - \delta_3$ . In Case (A) this gives a path  $\gamma$  in  $C$ , of length  $o(|z_0|)$ , joining  $z_3 = G_0(0)$  to  $z_2$  via  $z_0$ , and with  $|f'(z)| \leq c_2$  on  $\gamma$ . Since  $z_0$  is large so are  $z_2$  and  $z_3$ . Thus Lemma 5, applied at  $z_2$ , and integration of  $f'$  yield

$$f(z_3) = f(z_2) + o(|z_0|), \quad |f(z_3) - b_n z_3| \leq \varepsilon_2 |z_2| + o(|z_0|) \leq (\varepsilon_2 + o(1)) |z_3|. \quad (7)$$

But by the assumption of Case (A),  $f''$  has a zero in  $C$ , which must be at  $z_3$ , so that, by the hypotheses of the theorem,  $|f(z_3) - b_n z_3| \geq \kappa |z_3|$ . This contradicts (7), since  $\varepsilon_2$  is small. Next, in Case (B) the above analysis may be applied twice, to give a path  $\gamma$  in  $C$  of length  $o(|z_0|)$ , on which  $|f'(z)| \leq c_2$ , such that  $\gamma$  joins points  $w_p \in C \cap U_p$  and  $w_q \in C \cap U_q$  via  $z_0$ , where  $b_p$  and  $b_q$  are distinct and non-zero, and  $|f'(w_j) - b_j| < \varepsilon_1$  for  $j = p, q$ . Therefore the  $w_j$  satisfy  $w_j \sim z_0$  and  $|f(w_j) - b_j w_j| \leq \varepsilon_2 |w_j| \leq 2\varepsilon_2 |z_0|$  for  $j = p, q$ . Since  $\varepsilon_2$  is small and integration of  $f'$  along  $\gamma$  leads to  $f(w_p) - f(w_q) = o(|z_0|)$ , this case also delivers a contradiction.

It follows in both cases that  $|G'_0(u_0)| \geq c_3 |z_0|$ , which implies at once that  $|z_0 v'_0(z_0)| \leq c_4$ . Writing  $f'(z) = h_1^{-1}(v_0(z)^m)$  and using the fact that  $m \leq c_1$  and  $|f'(z_0)| < \tau$  gives  $|z_0 f''(z_0)| \leq c_5$ . To prove the last assertion requires a disc on which  $f'$  is univalent. To this end, observe that  $|G'_0(u_0)| \leq c_6 |z_0|$ , since  $z_0$  is large but  $C$  does not contain the point  $W_0$  chosen in Definition 1. Now choose  $u_0^*$  with  $|u_0^* - u_0| \leq \delta_4$  and  $|u_0^*| \geq \delta_4$ , and choose  $\delta_5$  so small that the function  $u \mapsto u^m$  is univalent on  $B(u_0^*, \delta_5)$ . Then Koebe's distortion theorem

implies that the image  $X_0$  of  $B(u_0^*, \delta_5)$  under  $G_0$  lies in  $B(z_0, \frac{1}{2}|z_0|) \cap C$  and contains a disc  $B(z_0^*, 2M_2|z_0^*|)$ , where  $z_0^* = G_0(u_0^*)$  and  $M_2 = \delta_6$ ; this requires only that  $\delta_4$  and  $\delta_5/\delta_4$  be small enough, independent of  $z_0$ . The function  $v_0(z)^m$  is univalent on  $X_0$  and therefore so is  $f'$ . Now take  $\zeta$  in  $B(z_0^*, M_2|z_0^*|)$  and set

$$g(z) = \frac{f'(\zeta + M_2|z_0^*|z) - f'(\zeta)}{M_2|z_0^*|f''(\zeta)} = z + \sum_{\mu=2}^{\infty} A_{\mu}z^{\mu}$$

for  $|z| < 1$ , so that the estimate (6) follows from Bieberbach's bound  $|A_2| \leq 2$ .  
□

It will be seen that hypothesis (i) of Theorem 3 plays a key role in the above proof of Lemma 7, principally by preventing  $z_0$  from lying too close to the boundary of  $C$ .

**Lemma 8** *With the notation of Lemma 5 and Definition 1, let  $z_1$  be large and satisfy*

$$z_1 \in U_p, \quad b_p \neq 0, \quad \sigma < |f'(z_1) - b_p| < \tau < s_1, \quad f'(z_1) \in J = \partial B_1, \quad (8)$$

and let  $C$  be the component of  $(f')^{-1}(B_1)$  with  $z_1 \in \partial C$ . Assume that one of the following two mutually exclusive conditions holds:

- (a) the function  $f'$  is not univalent on  $C$ ;
- (b) the function  $f'$  is univalent on  $C$ , and  $C \cap U_q$  is non-empty, for some  $q$  with  $0 \neq b_q \neq b_p$ .

Then there exists an open set  $H_1$ , with

$$H_1 \subseteq B\left(z_1, \frac{1}{2}|z_1|\right) \cap C \quad \text{and} \quad \partial H_1 \cap \partial C = \{z_1\}, \quad (9)$$

such that  $f'$  maps  $H_1$  onto an open disc  $K_1 \subseteq B_1$ , of diameter less than  $\rho$ , which is tangent to  $J = \partial B_1$  at  $f'(z_1)$ . Furthermore,  $H_1$  contains an open disc  $L_1$  of radius  $M_4|z_1|$  on which (6) holds; here both  $M_3$  and  $M_4$  are independent of  $z_1$  and  $C$ .

*Proof.* The component  $C$  is unique because  $z_1$  is large and  $f''$  has finitely many zeros which are not zeros of  $f'$ . As in Lemma 7 denote small positive constants by  $\delta_j$ , and positive constants by  $c_j$ ; these will again be independent of  $z_1$  and  $C$ . Let  $\gamma_0$  be the straight line segment

$$u = tu_1, \quad \delta_1 \leq t \leq 1, \quad u_1 = h_1(f'(z_1)) \in S(0, 1),$$

where  $\delta_1$  is chosen so small that  $|h_1(w)| \leq \delta_1$  implies that  $|w| \leq \delta_2 < \tau < \varepsilon_1$ . Using (8) and the conformal extension of  $h_1$  to  $B_1 \cup B(b_p, s_1)$  given by Lemma 6, define domains  $F_1 \subseteq B_1 \cup \{\zeta \in \mathbb{C} : \rho < |\zeta - b_p| < s_1\}$  and  $E_1$  by

$$E_1 = \{u \in \mathbb{C} : \text{dist}\{u, \gamma_0\} < \delta_3\} = h_1(F_1),$$

in which  $\delta_3$  is small compared to  $\delta_1$ , which ensures that  $0 \notin E_1$ . Then  $F_1$  contains no singular values of the inverse function  $(f')^{-1}$ , and  $z_1$  lies in a

component  $D$  of  $(f')^{-1}(F_1)$  such that  $h_1 \circ f'$  maps  $D$  conformally onto  $E_1$ . Let  $G_1 : E_1 \rightarrow D$  be the inverse function of  $h_1 \circ f'$ , and choose  $z_2 \in D$  with  $u_2 = h_1(f'(z_2)) = \delta_1 u_1$  and hence  $|f'(z_2)| \leq \delta_2 < \tau$ . Observe that  $z_2$  lies in  $C$ . Repeated application of the Koebe distortion theorem yields  $c_1|G'_1(u_1)| \leq |G'_1(u)| \leq c_2|G'_1(u_1)|$  on the line segment  $\gamma_0$ , and the image  $\sigma_1 = G_1(\gamma_0)$  is a path of length at most  $c_3|G'_1(u_1)|$  from  $z_1$  to  $z_2$  in  $D$ .

Suppose first that  $G'_1(u_1) = o(|z_1|)$ . Then  $z_2 \sim z_1$  and  $G'_1(u_2) = o(|z_1|)$ , from which it follows that  $z_2 f''(z_2)$  is large. Hence  $C$  satisfies neither condition (A) nor condition (B) of Lemma 7, and so cannot satisfy (b), because (b) implies (B) since  $C \cap U_p \neq \emptyset$  and  $b_p \neq 0$ . Hence  $f'$  is not univalent on  $C$  but  $C$  contains no zero of  $f''$ . Thus Lemma 4, with  $F = f'$ , shows that  $C$  must contain a path  $\Gamma$  tending to infinity on which  $f'(z)$  tends to 0, and  $C$  meets one of the components  $U_n$  with  $b_n = 0$ ; moreover,  $\log(h_1 \circ f')$  maps  $C$  univalently onto the left half plane. Therefore, since  $|h_1(f'(z_2))| \leq \delta_1$ , there exists a path  $\Gamma'$  in  $C$  joining  $z_2$  to some  $z_3 \in \Gamma$  on which  $|h_1(f'(z))| \leq \delta_1$  and  $|f'(z)| < \varepsilon_1$ , and hence  $z_2 \in U_n$ . Since  $z_1$  is large, and  $z_2 \sim z_1$ , Lemma 5 gives  $|f(z_2)| \leq \varepsilon_2|z_2|$  and  $|f(z_1) - b_p z_1| \leq \varepsilon_2|z_1|$ , in which  $b_p \neq 0$ . On the other hand  $|f'(z)| \leq c_4$  on  $\sigma_1$ , and so integration yields  $f(z_1) = f(z_2) + o(|z_1|)$  and a contradiction.

It must therefore be the case that  $|G'_1(u_1)| \geq c_5|z_1|$ . However, the point  $W_0$  chosen in Definition 1 does not lie in  $D$  and so  $|G'_1(u_1)| \leq c_6|z_1|$ . Now let  $G_2 = G_1 \circ h_1 : F_1 \rightarrow D$  be the inverse function of  $f'$ , and consider the point  $v_1 = f'(z_1) = h_1^{-1}(u_1) \in J$ . Then (8) yields  $c_7|z_1| \leq |G'_2(v_1)| \leq c_8|z_1|$ , as well as  $B(v_1, 2\delta_4) \subseteq F_1$  for some  $\delta_4 < \rho$ , and Koebe's distortion theorem gives  $c_9|z_1| \leq |G'_2(v)| \leq c_{10}|z_1|$  on  $B(v_1, \delta_4)$ . Hence  $G_2(B(v_1, \delta_5)) \subseteq B(z_1, \frac{1}{2}|z_1|)$ , provided  $\delta_5 \leq \delta_4$  is chosen small enough. Let  $K_1 \subseteq B(v_1, \delta_5) \cap B_1$  be an open disc of radius  $\delta_6 \leq \frac{1}{4}\delta_5$ , which is tangent to  $J$  at  $v_1$ . Then  $H_1 = G_2(K_1)$  satisfies (9), and  $H_1$  contains a disc  $B(z_1^*, 2M_4|z_1|)$ , with  $M_4 = \delta_7$ . It may now be assumed that  $M_3$  is large enough that (6) holds on  $L_1 = B(z_1^*, M_4|z_1|)$ , since  $f'$  is univalent on  $H_1$  and Bieberbach's theorem may be applied as in the proof of Lemma 7.  $\square$

## 5 The frequency of poles of $f$ and zeros of $f''$

**Lemma 9** *Let  $w_1, \dots, w_Q$  be pairwise distinct poles of  $f$  with  $|w_j|$  large. For  $1 \leq j \leq Q$  let  $D_j$  be the component of  $(f')^{-1}(B_2)$  in which  $w_j$  lies. Then for each  $j$  there exists  $p_j \in \mathbb{Z}$  such that  $\partial D_j$  contains a Jordan arc  $\lambda_j$  which is mapped univalently by  $f'$  onto a line segment  $\mu_j$  of length at least  $\sigma$ , and these may be chosen so that*

$$\lambda_j \subseteq U_{p_j}, \quad \mu_j \subseteq \{\zeta \in J = \partial B_2 : \sigma < |\zeta - b_{p_j}| < \tau\}, \quad b_{p_j} \neq 0, \quad (10)$$

where  $U_{p_j}$  and  $b_{p_j}$  are as in Lemma 5, while  $\sigma$  and  $\tau$  are as in Definition 1.

Moreover, if points  $z_j$  are chosen such that  $z_j \in \lambda_j$  for  $1 \leq j \leq Q$ , then each  $|z_j|$  is large and for each  $j$  there exists an open disc  $L_j \subseteq B(z_j, \frac{1}{2}|z_j|)$  of

radius  $M_4|z_j|$ , on which (6) holds, where  $M_4$  is as in Lemma 8. The  $L_j$  are pairwise disjoint.

*Proof.* By the discussion in Section 3, each  $D_j$  is unbounded and simply connected and the boundary  $\partial D_j$  contains no zeros of  $f''$ . Each component of  $\partial D_j$  is a simple path tending to infinity in both directions, and there exists a component  $\Gamma_j$  of  $\partial D_j$  which separates  $w_j$  from the point  $W_0$  chosen in Definition 1. Since  $D_j$  contains a pole of  $f$  it follows that  $f'$  is finite-valent on  $D_j$ . Thus as  $z$  tends to infinity in either direction along  $\Gamma_j$  the image  $f'(z)$  must tend to a non-zero finite asymptotic value of  $f'$ . In particular,  $\Gamma_j$  meets some  $U_p$  as in Lemma 5 with  $b_p \neq 0$ , and following  $\Gamma_j$  while staying in  $U_p$  gives  $\lambda_j$  and  $\mu_j$  as in (10). Furthermore, each  $w_j$  is large and, for any  $M_5 > 0$ , the disc  $B(0, M_5)$  meets only finitely many components of  $(f')^{-1}(B_2)$ , each of which contains at most one pole of  $f$ . Hence if  $z_j \in \lambda_j$  then  $z_j$  is large.

To prove the existence of the  $L_j$ , choose for each  $j$  a component  $E_j$  of  $(f')^{-1}(B_1)$  with  $\Gamma_j \subseteq \partial E_j$ . Since  $\Gamma_j$  separates the pole  $w_j$  of  $f$  from  $W_0$  it follows that  $\Gamma_j$  is not the whole boundary  $\partial E_j$ . In particular, if  $f'$  is univalent on  $E_j$  then  $\Gamma_j$  must meet components  $U_p$  and  $U_q$  with  $b_p$  and  $b_q$  distinct and non-zero. Thus each of these components  $E_j$  of  $(f')^{-1}(B_1)$  satisfies one of the conditions (a), (b) of Lemma 8, which may now be applied with  $z_1$  replaced by each  $z_j$ . This gives open sets  $H_j \subseteq B(z_j, \frac{1}{2}|z_j|) \cap E_j$ , each containing an open disc  $L_j$  of radius  $M_4|z_j|$  on which (6) holds. Moreover,  $f'$  maps  $H_j$  onto a disc  $K_j \subseteq B_1$  which is tangent to  $J$  at  $f'(z_j)$  and has diameter less than  $\rho$ .

To show that the  $L_j$  are disjoint, suppose that  $1 \leq j < j' \leq Q$  and that  $H_j \cap H_{j'} \neq \emptyset$ , from which it follows of course that  $K_j \cap K_{j'} \neq \emptyset$ . Since  $\rho$  is small compared to  $\sigma$  and  $z_j \in \lambda_j$ , the open disc  $U = B(f'(z_j), 3\rho)$  contains no singular value of  $(f')^{-1}$ , by (10). But  $K_j$  and  $K_{j'}$  have diameter less than  $\rho$ , and so their closures lie in  $U$ . Thus  $H_j$  and  $H_{j'}$  both lie in the same component of  $(f')^{-1}(U)$ , as do  $z_j$  and  $z_{j'}$ , which forces  $\Gamma_j = \Gamma_{j'}$  and gives a contradiction.  $\square$

**Lemma 10** *Let  $L(r) \rightarrow \infty$  with  $L(r) \leq \frac{1}{8} \log r$  as  $r \rightarrow \infty$ , and for  $k > 0$  and large  $r$  define the annulus  $A(k)$  by  $A(k) = \{z \in \mathbb{C} : re^{-kL(r)} \leq |z| \leq re^{kL(r)}\}$ . Then the number  $N_1$  of distinct poles of  $f$  and zeros of  $f''$  in  $A(1)$  satisfies*

$$N_1 = O(\phi(r)) \quad \text{as } r \rightarrow \infty, \text{ where } \phi(r) = L(r) + \frac{\log r}{L(r)}. \quad (11)$$

*Proof.* Assume that  $r$  is large and that  $A(1)$  contains  $Q = 2N$  distinct poles  $w_1, \dots, w_{2N}$  of  $f$ , with  $\phi(r) = o(N)$ . For  $j = 1, \dots, Q$  let  $D_j$  be the component of  $(f')^{-1}(B_2)$  in which  $w_j$  lies; let  $q_j$  be the multiplicity of the pole of  $f'$  at  $w_j$ . Each  $D_j$  is unbounded and simply connected and may be assumed not to contain the origin. Let  $v_j = (h_2 \circ f')^{1/q_j}$ , so that  $v_j$  maps  $D_j$  conformally onto  $B(0, 1)$ , with  $v_j(w_j) = 0$ .

For  $0 < t < \infty$  let  $\theta_j(t)$  be the angular measure of  $D_j \cap S(0, t)$ . Let  $c$  denote positive constants, not necessarily the same at each occurrence, but

not depending on  $r, L(r)$  or  $N$ . For  $m \in \mathbb{N}$  the Cauchy-Schwarz inequality gives  $m^2 \leq 2\pi \sum_{j=1}^m 1/\theta_j(t)$  so that, as in [8], at least  $N$  of the  $D_j$  have

$$\int_{2re^{L(r)}}^{(1/2)re^{2L(r)}} \frac{dt}{t\theta_j(t)} > cNL(r), \quad \int_{2re^{-2L(r)}}^{(1/2)re^{-L(r)}} \frac{dt}{t\theta_j(t)} > cNL(r). \quad (12)$$

It may be assumed after re-labelling if necessary that (12) holds for  $D_1, \dots, D_N$ . Since  $w_j$  lies in  $A(1)$ , it follows from Lemma 1 that, with  $\sigma_j = \partial D_j \setminus A(2)$ ,

$$\begin{aligned} \omega(w_j, \sigma_j, D_j) &\leq c \exp \left( -\pi \int_{2re^{L(r)}}^{(1/2)re^{2L(r)}} \frac{dt}{t\theta_j(t)} \right) \\ &\quad + c \exp \left( -\pi \int_{2re^{-2L(r)}}^{(1/2)re^{-L(r)}} \frac{dt}{t\theta_j(t)} \right). \end{aligned}$$

Combining this with (11), (12) and condition (iii) of the theorem shows that  $\omega(w_j, \sigma_j, D_j) = o(1/q_j)$  for  $j = 1, \dots, N$ . But Lemma 9 gives an arc  $\lambda_j \subseteq \partial D_j$ , mapped by  $f'$  onto a line segment  $\mu_j \subseteq J$  as in (10), of length at least  $\sigma$ . Since  $b_{p_j}$  in (10) is not a vertex of  $J$ , while  $\tau$  is small, an application of the Schwarz reflection principle to  $h_2$  shows that  $h_2 \circ f'$  maps  $\lambda_j$  to an arc of  $S(0, 1)$  of length at least  $c$ , and  $v_j(\lambda_j)$  has angular measure at least  $c/q_j$ . The conformal invariance of harmonic measure under  $v_j$  implies that  $\lambda_j$  cannot be contained in  $\sigma_j$ , and so there exists  $z_j \in \lambda_j \cap A(2)$ . The corresponding  $N$  pairwise disjoint discs  $L_j$  given by Lemma 9 lie in the annulus  $A(3)$ , and hence

$$cN \leq \sum_{j=1}^N \int_{L_j} |z|^{-2} dx dy \leq \int_{A(3)} |z|^{-2} dx dy \leq cL(r) \leq c\phi(r) = o(N).$$

This is a contradiction and the asserted upper bound for the number of distinct poles in  $A(1)$  is proved. The same upper bound for the number of distinct zeros  $\zeta_j$  of  $f''$  in  $A(1)$  follows at once from Lemma 7, because such zeros give rise to pairwise disjoint discs  $B(\zeta_j^*, M_2|\zeta_j^*|) \subseteq A(2)$ .  $\square$

Since all but finitely many zeros of  $f''$  are zeros of  $f'$ , which have bounded multiplicities by assumption, choosing  $L(r) = \frac{1}{8} \log r$  in Lemma 10 gives

$$\bar{n}(r^{9/8}, f) - \bar{n}(r^{7/8}, f) + n(r^{9/8}, 1/f'') - n(r^{7/8}, 1/f'') = O(\log r),$$

and so

$$\bar{N}(r, f) + N(r, 1/f'') = O(\log r)^2 \quad \text{as } r \rightarrow \infty. \quad (13)$$

**Lemma 11** *The lower order of  $f''/f'$  is at least  $\frac{1}{2}$ .*

*Proof.* If this is not the case then the function  $f'/f''$  has finitely many poles and is transcendental of lower order less than  $\frac{1}{2}$ . The  $\cos \pi\lambda$  theorem [1] now

gives  $r_j \rightarrow +\infty$  such that  $f''(z)/f'(z) = O(r_j^{-2})$  on  $S(0, r_j)$ . Moreover, the main result of [10] gives a path  $\gamma$  tending to infinity with

$$\int_{\gamma} \left| \frac{f''(z)}{f'(z)} \right| |dz| < \infty.$$

This implies that, as  $z$  tends to infinity in the union of  $\gamma$  and the  $S(0, r_j)$ , the image  $f'(z)$  tends to some  $b_n$  as in Lemma 5, contradicting the fact that the  $U_n$  are simply connected.  $\square$

**Lemma 12** *The function  $f''$  has the form  $f'' = \Pi_1/\Pi_2$ , where  $\Pi_1$  and  $\Pi_2$  are entire such that  $\Pi_2$  has finite order and  $\Pi_1 \not\equiv 0$  has order 0. Moreover, the lower order of  $\Pi_2$  is at least  $1/2$ .*

*Proof.* Using (1) and (13) shows that  $N(r, f'')$  has finite order and  $N(r, 1/f'')$  has order 0. Since  $f''$  has finite lower order, this gives the asserted representation for  $f''$ . On the other hand, Lemma 11 implies that  $f'$  has lower order at least  $1/2$  and so has  $f''$ , and hence so has  $\Pi_2$ .  $\square$

**Lemma 13** *Let  $h(z) = zf'''(z)/f''(z)$ . For all  $s \geq 1$  lying outside a set  $E_0$  of finite logarithmic measure, there exists  $\zeta_s$  with  $|\zeta_s| = s$  and  $|h(\zeta_s)| > s^{1/3}$ .*

*Proof.* Take  $\Pi_1$  and  $\Pi_2$  as in Lemma 12. Applying the Wiman-Valiron theory [5, Theorem 12] and standard estimates for logarithmic derivatives [3] makes it possible to write, for  $|\zeta_s| = s$  with  $|\Pi_2(\zeta_s)| = M(s, \Pi_2)$  and  $s$  outside a set of finite logarithmic measure,

$$\frac{f'''}{f''} = \frac{\Pi_1'}{\Pi_1} - \frac{\Pi_2'}{\Pi_2}, \quad \left| \frac{\Pi_2'(\zeta_s)}{\Pi_2(\zeta_s)} \right| \sim \frac{\nu(s)}{s}, \quad \left| \frac{\Pi_1'(\zeta_s)}{\Pi_1(\zeta_s)} \right| \leq s^{-3/4}.$$

Here  $\nu(s)$  is the central index of  $\Pi_2$  and, by standard facts from the Wiman-Valiron theory [5], the lower order of  $\nu(s)$  is the same as that of  $\log M(s, \Pi_2)$ , and so is at least  $1/2$ .  $\square$

## 6 Completion of the proof of Theorem 3

Lemma 12 shows that  $f$  has finite order  $\rho(f)$ . Thus it remains only to prove that  $f$  has finitely many poles and  $f''$  has finitely many zeros, so assume that this is not the case. Lemmas 7 and 9 give a positive real number  $d_1$  and  $w \in \mathbb{C}$  with  $|w| = r$  arbitrarily large, such that (6) holds on the disc  $B(w, d_1 r)$ . Let  $\varepsilon$  and  $K$  be positive, with  $\varepsilon$  small, and let

$$U_K = \left\{ z \in \mathbb{C} : \frac{1}{K} < |z| < K, \quad |z - 1| > d_1 \right\}.$$

Here  $K$  is chosen so large that the harmonic measure with respect to  $U_K$  satisfies

$$\omega(z, S(0, 1/K) \cup S(0, K), U_K) < \varepsilon \quad \text{for } z \in U_K, \quad \frac{1}{2} < |z| < 2. \quad (14)$$

Denote by  $d_j$  positive constants which are independent of  $r, \varepsilon$  and  $K$ . Standard estimates from [3] give a real number  $S = S_r$  such that

$$K < S < 2K \quad \text{and} \quad |h(z)| \leq |z|^{d_2} \quad \text{for } |z| = \frac{r}{S} \quad \text{and} \quad |z| = rS, \quad (15)$$

in which  $h(z) = zf'''(z)/f''(z)$  as in Lemma 13 and  $d_2 = \rho(f) + 1$ . Let  $w_1, \dots, w_Q$  be the poles of  $h$  in  $r/S \leq |z| \leq rS$ . Applying Lemma 10 with  $L(r) = (\log r)^{1/2}$  shows that  $Q \leq d_3(\log r)^{1/2}$ .

On the annulus  $A$  given by  $r/S \leq |z| \leq rS$  set

$$u(z) = \log |h(z)| - \log M_3 + \sum_{1 \leq j \leq Q} \log \frac{|z - w_j|}{4Kr} \leq \log |h(z)| - \log M_3, \quad (16)$$

where  $M_3$  is as in (6) and may be assumed to be at least 1, and the sum is empty if there are no poles  $w_j$ . Then  $u$  is subharmonic on  $A$ , with  $u(z) \leq 0$  on the closure of  $B(w, d_1 r)$  by (6), and

$$u(z) \leq \log |h(z)| \leq d_2 \log |z| \leq d_2 \log(2Kr) \quad \text{for } z \in S(0, r/S) \cup S(0, rS),$$

by (15). Hence (14) and the monotonicity of harmonic measure yield

$$u(z) \leq \varepsilon d_2 \log(2Kr) \quad \text{for } \frac{r}{2} < |z| < 2r. \quad (17)$$

Now Lemma 2 shows that (4) holds, with  $\Lambda = r/24$ , for all  $z$  outside a union  $P_r$  of discs having sum of radii at most  $r/4$ . Choose  $s \in (r/2, 2r) \setminus E_0$ , with  $E_0$  as in Lemma 13, such that the circle  $S(0, s)$  does not meet  $P_r$ . Thus Lemma 13 and (17) give rise to  $\zeta_s \in S(0, s)$  such that

$$\begin{aligned} \frac{1}{3} \log s &\leq \log |h(\zeta_s)| \\ &\leq \varepsilon d_2 \log(2Kr) + \log M_3 + \sum_{1 \leq j \leq Q} \log \frac{4Kr}{|\zeta_s - w_j|} \\ &\leq \varepsilon d_2 \log(2Kr) + \log M_3 + Q \log(96K) \\ &\leq \varepsilon(\rho(f) + 1) \log(4Ks) + \log M_3 + d_3(\log 2s)^{1/2} \log(96K). \end{aligned}$$

Since  $\varepsilon$  may be chosen arbitrarily small, while  $s$  is large, this gives a contradiction and the proof of Theorem 3 is complete.  $\square$

*Remark.* Hypothesis (iii) on the multiplicities of poles may not really be essential for Theorem 3 but it does play a key role in the above proof. If it is assumed merely that  $f$  has finite lower order, then techniques such as Pólya peaks should give annuli on which the analysis of Lemma 10 can be applied,

but it seems difficult to ensure that these contain enough distinct poles of  $f$  to prevent the discs on which (6) holds from being so remote that the method of Section 6 fails.

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