CRITICAL VALUES OF SLOWLY GROWING MEROMORPHIC FUNCTIONS

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Abstract. Let f be transcendental and meromorphic in the plane. We obtain sharp lower bounds for the growth of f, in terms of the minimum spherical distance between the critical values of f. The extremal examples arise from elliptic functions.

1. Introduction

For a function f transcendental and meromorphic in the plane, the critical values are those values taken by f at critical points of f, that is, multiple poles of f and zeros of f'. Together with the asymptotic values of f, values a such that $f(z) \to a$ as $z \to \infty$ along a path γ_a, the critical values play an important role in iteration theory [3]. Our starting point is the following theorem, in which

$$\underline{L}(f) = \liminf_{r\to\infty} \frac{T(r,f)}{(\log r)^2}, \quad \overline{L}(f) = \limsup_{r\to\infty} \frac{T(r,f)}{(\log r)^2},$$

where $T(r, f)$ denotes the Nevanlinna characteristic [8].

Theorem 1.1 ([12]). *Let f be transcendental and meromorphic in the plane, with $\underline{L}(f) = 0$. Then f has infinitely many critical values.*

The same conclusion had earlier been proved for $\overline{L}(f) = 0$ in [11]. Theorem 1.1 is essentially sharp because of an example, with $\overline{L}(f) < \infty$ and four critical values, constructed by Bank and Kaufman [2, 10] from the Weierstrass doubly periodic function (see §2).

For entire functions the sharp condition is [5]

$$\liminf_{r\to\infty} \frac{T(r,f)}{\sqrt{r}} = 0. \tag{1}$$

If f is transcendental entire and satisfies (1) then ∞ is a limit point of critical values of f, as may be seen [5] by applying the $\cos\pi\lambda$ theorem [9, Theorem 6.7] and the argument of [16, p.287]. Examples such as $\cos\mu\sqrt{z}$, with $\mu > 0$, show that this result is sharp.

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The present investigations arose from the following question: is there a lower bound for $\underline{L}(f)$ in terms of the number of critical values of f? In this direction we shall say that a critical point ζ of f lies over the set E if $f(\zeta) \in E$, and we note that if f is transcendental and meromorphic in the plane such that all but finitely many critical points of f lie over a given set of two values, then [6] the order of f is at least $\frac{1}{2}$, this result sharp because of the examples $\cos \mu\sqrt{z}$ and $\tan^2 \sqrt{z}$. In Example 2.1 we modify the Bank-Kaufman construction to obtain a transcendental meromorphic f with $\overline{L}(f)$ finite and just three critical values.

Theorem 1.2. *There exists a positive constant B with the following property. Let a_1, a_2, a_3 be distinct elements of the extended complex plane. If f is transcendental and meromorphic in the plane such that all but finitely many critical points of f lie over $\{a_1, a_2, a_3\}$, then $\underline{L}(f) \geq B$.*

Next, in Example 2.2 we show that there exist transcendental meromorphic f, with $\underline{L}(f)$ arbitrarily small, and with just four critical values. In these examples, however, the minimum spherical distance between the critical values tends to 0 as $\underline{L}(f)$ tends to 0. We denote by

$$q(z,w) = \frac{|z-w|}{\sqrt{(1+|z|^2)(1+|w|^2)}}, \quad q(z,\infty) = \frac{1}{\sqrt{(1+|z|^2)}},$$

the chordal distance between elements of the extended plane.

Theorem 1.3. *There exists a positive absolute constant c_0 with the following property. Let $\{a_1, \ldots, a_N\}$ be a finite set of distinct elements of the extended complex plane. Let*

$$\alpha = \sup\{t > 0 : \exists a \in \mathbb{C} \cup \{\infty\},\, q(a, a_j) > t,\, j = 1, \ldots, N\}. \tag{2}$$

Let f be a function transcendental and meromorphic in the plane such that all but finitely many critical points of f lie over $\{a_1, \ldots, a_N\}$. Then

$$\exp\left(\frac{-1}{4\underline{L}(f)}\right) \geq c_0 \alpha \min\{q(a_j, a_{j'}) : j \neq j'\}. \tag{3}$$

It is obvious that in Theorem 1.3 we may take $\alpha \geq b/\sqrt{N}$, for some positive absolute constant b. Hence (3) may be replaced by

$$\sqrt{N} \exp\left(\frac{-1}{4\underline{L}(f)}\right) \geq c_1 \min\{q(a_j, a_{j'}) : j \neq j'\}.$$

Thus Theorem 1.3 shows that if $\underline{L}(f)$ is small then either the number of critical values must be large or the minimum spherical distance between them must be small.

The function of Example 2.2 has $\underline{L}(f) = \delta$ arbitrarily small, and all its critical points over a set $\{a_1, \ldots, a_4\}$, where

$$\min\{q(a_j, a_{j'}) : j \neq j'\} \sim 4e^{-1/4\delta}, \quad \delta \to 0.$$

Further, in these examples $\alpha \geq 1/\sqrt{2} - o(1)$ as $\delta \to 0$, and so Theorem 1.3 is essentially sharp.

Note that Theorem 1.2 follows at once from Theorem 1.3 and Nevanlinna's first fundamental theorem, since we may apply a Möbius map sending a_1, a_2, a_3 to $0, 1, \infty$.

We remark finally that the proof of Theorem 1.1 in [12] depended on representations $f(z) = cz^N(1+o(1))$ in annuli, using a result from [15]. The present approach is simpler and more geometric.

2. Functions of small growth with three or four critical values

We modify the construction of Bank and Kaufman [2] (see also [10, p.234]), starting with the Weierstrass doubly periodic function $\wp(z)$, with primitive periods

$$\omega_1 = P, \quad \omega_2 = 2\pi i = \tau\omega_1, \quad \text{Re}(P) > 0, \quad \text{Im}(\tau) > 0. \tag{4}$$

The function $w(z) = \wp(\log z)$ is then meromorphic in $0 < |z| < \infty$. Next, set $\omega_3 = \omega_1 + \omega_2$ and $e_j = \wp(\omega_j/2)$, and

$$u(\zeta) = w(z), \quad z + 1/z = \zeta. \tag{5}$$

Since $\wp$ is even, (5) defines u as meromorphic on $\mathbb{C} \setminus \{-2, 2\}$. Further,

$$\lim_{\zeta \to 2} u(\zeta) = \lim_{z \to 1} \wp(\log z) = \infty, \quad \lim_{\zeta \to -2} u(\zeta) = \lim_{z \to -1} \wp(\log z) = e_2, \tag{6}$$

so that u extends to be meromorphic on $\mathbb{C}$. The computation of [10, p.235] gives

$$(\zeta^2 - 4)u'(\zeta)^2 = \wp'(\log z)^2 = 4(u(\zeta) - e_1)(u(\zeta) - e_2)(u(\zeta) - e_3) \tag{7}$$

so that the only critical values of u are ∞, e_1, e_2, e_3. Dividing (7) through by $(u - e_1)(u - e_2)$, we get $m(r, u) = S(r, u)$. Also $w(z) = \infty$ at the points $\log z = mP + n2\pi i$, with m, n integers, and so u has poles at the points

$$\zeta_m = e^{mP} + e^{-mP}, \quad m \in \mathbb{Z}.$$

Using (4), we have $|\zeta_m| \to \infty$ and $|\zeta_{m+1}/\zeta_m| \to e^{\text{Re}(P)}$ as $m \to +\infty$, so that

$$T(r, u) \sim N(r, u) \sim (2\delta + o(1))(\log r)^2, \quad r \to \infty, \quad \delta = \frac{1}{2\text{Re}(P)}. \tag{8}$$

We will use some basic facts from the theory of the elliptic modular function [1, Chapter 3], in which

$$\rho(\tau) = \frac{e_3 - e_1}{e_2 - e_1}, \quad \tau = \frac{2\pi i}{P}. \tag{9}$$

Example 2.1.

Here we construct an example with just three critical values. Recall [1] that $\rho(\tau)$ maps the upper half plane $\text{Im}(\tau) > 0$ onto $\mathbb{C} \setminus \{0, 1\}$. Choose τ so that $\rho(\tau) = \eta + 1 = e^{2\pi i/3} + 1$. Since $e_1 + e_2 + e_3 = 0$ and $\eta^2 = -\eta - 1$

we get $e_1^3 = e_2^3 = e_3^3$. Thus $v(\zeta) = u(\zeta)^3$ has critical values $0, \infty, e_1^3$, and (8) gives $\overline{L}(v) < \infty$.

Example 2.2.

We construct next an example with four critical values, and with $\overline{L}(f)$ arbitrarily small. Let P be real and positive. Then (8) gives

$$\underline{L}(u) = \overline{L}(u) = 1/P = 2\delta. \tag{10}$$

By (4) and [1, pp.43-6] we have, as $P \to \infty$,

$$\tau \to 0, \quad \rho(\tau) = 1 - \rho(-1/\tau) \sim 16e^{-P/2} = 16e^{-1/4\delta}. \tag{11}$$

By (6) and (7), all roots of $u(\zeta) = e_1, e_3$ are double, and the function

$$f(\zeta) = \left(\frac{u(\zeta) - e_3}{u(\zeta) - e_1} \right)^{\frac{1}{2}}$$

is meromorphic, with $\underline{L}(f) = \overline{L}(f) = \delta$ by (10). Finally, f has critical values ± 1 and $\pm e_4$, in which

$$e_4 = \left(\frac{e_2 - e_3}{e_2 - e_1} \right)^{1/2} = (1 - \rho(\tau))^{1/2}, \quad |e_4 - 1| \sim 8e^{-1/4\delta}, \quad \delta \to 0,$$

using (9) and (11). Thus the exponent in (3) is sharp.

3. Lemmas needed for Theorem 1.3

For $a \in \mathbb{C}$ and $r > 0$ set

$$C(a,r) = \{z \in \mathbb{C} : |z - a| = r\}, \quad B(a,r) = \{z \in \mathbb{C} : |z - a| < r\},$$

and for $K > 1$ denote by $A(r,K)$ the annulus

$$A(r,K) = \{z \in \mathbb{C} : r/K < |z| < Kr\}.$$

Lemma 3.1. *Let f be transcendental and meromorphic in the plane, and let $a \in \mathbb{C} \cup \{\infty\}$ and $P \in (0,1)$ be such that f has at most finitely many critical points ζ with $q(a, f(\zeta)) \le P$, and no asymptotic values w with $q(a,w) \le P$. Set $X = \{z \in \mathbb{C} : q(a, f(z)) < P\}$ and let Y be a component of X. Then:*
(i) Y is bounded;
(ii) if $Y^* = \{z \in Y : 0 < q(a, f(z)) < P\}$ contains no critical points of f then Y is simply connected.

Next, let $|z^*| = r$ be large, with $q(a, f(z^*)) < P/2$, and let L be that component of the set $\{z \in \mathbb{C} : q(a, f(z)) < P/2\}$ which contains z^*. Then:
(iii) f maps L onto $\{w : q(a,w) < P/2\}$;
(iv) L lies in the annulus $A(r, d_1)$, where $d_1 = e^{8\pi}$.

Proof. The proof of assertions (i) and (ii) of Lemma 3.1 is a modified version of the argument of [16, p.287] (see also [13, pp.459-461]). To this end, assume without loss of generality that $a = \infty$, and let $Q = \sqrt{1/P^2 - 1}$ and

$$W = \{w \in \mathbb{C} \cup \{\infty\} : Q < |w| \le \infty\}.$$

Then $X = f^{-1}(W)$. Next, let $b_1, \ldots, b_m$ be the critical values of f in $Q < |w| < \infty$, and set

$$V = W \setminus \bigcup_{j=1}^{m} \{te^{i \arg b_j} : Q \leq t \leq |b_j|\}.$$

Let C be a component of the set $f^{-1}(W)$, let D be a component of $f^{-1}(V)$ with $D \subseteq C$, and let $H = \{u \in \mathbb{C} : e^u \in V\}$. Take $z_0 \in D$, with $f(z_0)$ finite, and take that branch of the inverse function f^{-1} mapping $w_0 = f(z_0)$ to z_0. Choose $u_0 \in H$ with $e^{u_0} = w_0$. Then the function $h(u) = f^{-1}(e^u)$ extends by the monodromy theorem to an analytic function on H, and $h(H) = D^* = \{z \in D : |f(z)| < \infty\}$ [13, p.459].

If h is univalent on H then the image of the line $\mathrm{Re}(u) = M$, with M large and positive, is a level curve of f, tending to infinity in both directions, so that $h(u) \to \infty$ as $u \to \infty, u \in \mathbb{R}$. But this implies that ∞ is an asymptotic value of f, which we have excluded by hypothesis.

Thus h is not univalent on H and a standard argument [16, p.287] shows that h has period $k2\pi i$, for some minimal positive integer k. With M again large and positive, the function $h(k \log \zeta) = f^{-1}(\zeta^k)$ is analytic and univalent on $\infty > |\zeta| > M$, and has a limit as $\zeta \to \infty$, which must be finite since f is transcendental. Let v lie on the finite boundary ∂H of H. Continuing h along the boundary arcs of H and using the Phragmén-Lindelöf principle shows that $\limsup_{u \to v, u \in H \cup \partial H} |h(u)| < \infty$. Thus $h(H) = D^* = \{z \in D : |f(z)| < \infty\}$ is bounded, and so is D.

Suppose now that C is unbounded. Then C must contain infinitely many components D_n of the set $f^{-1}(V)$. However, if the closures E_m, E_n of two distinct components D_m, D_n have non-empty intersection, then E_m, E_n must share a critical point of f, and there are only finitely many such critical points available. This proves assertion (i) of Lemma 3.1.

Assertion (ii) is proved in the same way, but taking $V = W$. In this case $h(k \log \zeta) = f^{-1}(\zeta^k)$ is univalent on $|\zeta| > Q^{1/k}$.

To prove assertions (iii) and (iv) we apply a logarithmic change of variables used in [4, 7] and elsewhere. We know that all components of the set $\{z \in \mathbb{C} : q(\infty, f(z)) < P\}$ are bounded, and that all but finitely many of them are simply connected. Hence, if z^* and L are as in the hypotheses, we may assume that if L_j is that component of the set $\{z \in \mathbb{C} : q(\infty, f(z)) < P\}$ which contains L, then L_j is simply connected and does not contain the origin, and $g = 1/f$ maps L_j conformally onto the disc $B(0, 1/Q)$, from which (iii) follows at once. Let G be that branch of the inverse function g^{-1} mapping $B(0, 1/Q)$ onto L_j. Thus

$$L = G(B(0, 1/Q')), \quad Q' = \sqrt{4/P^2 - 1}. \tag{12}$$

Further, $\phi = \log G$ is defined and univalent on $B(0, 1/Q)$. Since $G = e^{\phi}$ is univalent on $B(0, 1/Q)$, it follows that ϕ maps $B(0, 1/Q)$ onto a region containing no disc of radius greater than π. Hence Koebe's one-quarter theorem gives, for w, w_1, w_2 in $B(0, 1/Q')$,

$$|\phi'(w)| \le \frac{4\pi}{1/Q - |w|}, \quad \left|\log \left|\frac{G(w_1)}{G(w_2)}\right|\right| \le \frac{8\pi/Q'}{1/Q - 1/Q'} = \frac{8\pi}{Q'/Q - 1}.$$

Since

$$\frac{Q'}{Q} = \sqrt{\frac{4 - P^2}{1 - P^2}} \ge 2,$$

assertion (iv) follows from (12). □

Our next lemma uses an idea which goes back at least to Valiron [17], but which does not seem to suffice to get the right exponent on the left-hand side of (3). It does, however, allow us to dispose of the case where f has an asymptotic value.

Lemma 3.2. *Let f be transcendental and meromorphic in the plane, with at least one asymptotic value, and assume that $\underline{L}(f)$ is finite. Then f has infinitely many critical values.*

Proof. Assume that f is as in the hypotheses, but with finitely many critical values. We may assume that $f(z) \to 0$ as $z \to \infty$ along a path $\gamma \to \infty$. Since $\underline{L}(f)$ is finite we have

$$\liminf_{r \to \infty} \frac{n(r, f) + n(r, 1/f) + n(r, 1/(f-1))}{\log r} < \infty.$$

Thus there exist a positive constant M and a sequence $r_n \to \infty$ such that f omits the values $0, 1, \infty$ on the annuli $A(r_n, e^{2M})$. Then the family of functions $f_n(z) = f(r_n z)$ is normal on $A(1, e^{2M})$ by Montel's theorem, and since $f(z) \to 0$ as $z \to \infty$ on γ there exists a positive constant M_1, independent of n, such that $|f(z)| \le M_1$ for $r_n e^{-M} \le |z| \le r_n e^M$. An application of the two-constants theorem for subharmonic functions [16] now shows that $f(z) \to 0$ uniformly on the union of the circles $C(0, r_n)$. Thus 0 is the only asymptotic value of f. Take a positive ε, so small that f has no critical values in $0 < |w| < 2\varepsilon$. Then for large n, the circle $C(0, r_n)$ lies in a simply connected component of the set $\{z \in \mathbb{C} : |f(z)| < \varepsilon\}$, by Lemma 3.1, and this is plainly impossible. □

Lemma 3.3. *There exists a positive constant d_2 with the following property. Let $\delta > 0$ and let f be transcendental and meromorphic in the plane, with $\underline{L}(f) < \delta$ and with no asymptotic values. Let $a \in \mathbb{C} \cup \{\infty\}$ and let $\sigma \in (0, 1)$ be such that all but finitely many critical points z of f have $q(a, f(z)) > \sigma$. Then there exists a sequence $r_n \to \infty$ such that the image of the circle $C(0, r_n)$ under f has spherical diameter at most $d_2 \sigma^{-1} e^{-1/4\delta}$.*

Proof. We may obviously assume without loss of generality that δ is small and $a = \infty$. Since $N(r, f) \le T(r, f)$ and $\underline{L}(f) < \delta$, there exists a sequence $r_n \to \infty$ such that f has no poles in the annuli $A(r_n, K)$, where $K = e^{1/4\delta}$ is large. Hence Lemma 3.1 gives, for large n,

$$q(\infty, f(z)) \ge \sigma/2, \quad z \in A(r_n, K/d_1), \quad d_1 = e^{8\pi}.$$

Thus, denoting positive absolute constants by d_j, we have

$$|f(z)| \leq d_3\sigma^{-1}, \quad s_n = r_n d_1/K \leq |z| \leq t_n = r_n K/d_1,$$

and we estimate the diameter of the image of $C(0, r_n)$ as in [14, Lemma 1]. Cauchy's integral formula gives, for $|w| = r_n$,

$$|f'(w)| = \left| \frac{1}{2\pi i} \int_{|z|=t_n} \frac{f(z)}{(z-w)^2} dz - \frac{1}{2\pi i} \int_{|z|=s_n} \frac{f(z)}{(z-w)^2} dz \right| \leq \frac{d_4}{\sigma K r_n},$$

and the spherical diameter of the image of $C(0, r_n)$ is then at most

$$\int_{|z|=r_n} |f'(w)| \, |dw| \leq \frac{2\pi d_4}{\sigma K}.$$

□

4. Proof of Theorem 1.3

Let $a_1, \ldots, a_N$ be distinct elements of the extended complex plane and define α by (2). Choose $a \in \mathbb{C} \cup \{\infty\}$ with $q(a, a_j) > \sigma = \alpha/2$ for all j.

Let positive constants δ and ε be defined by

$$\varepsilon = \min\{q(a_j, a_{j'}) : j \neq j'\}, \quad e^{-1/4\delta} = \frac{\sigma\varepsilon}{32 d_2} = \frac{\alpha\varepsilon}{64 d_2},$$

in which d_2 is the positive constant of Lemma 3.3. Let f be transcendental and meromorphic in the plane, such that all but finitely many critical points of f lie over the set $\{a_1, \ldots, a_N\}$, and assume that $\underline{L}(f) < \delta$. By Lemma 3.2 we may assume that f has no asymptotic values.

Lemma 3.3 now gives a sequence $r_n \to \infty$ such that the image under f of each circle $C(0, r_n)$ has spherical diameter at most $\varepsilon/32$. We may assume that $f(r_n) \to A \in \mathbb{C} \cup \{\infty\}$ as $n \to \infty$. Now the open spherical disc of centre A and radius $\varepsilon/2$ contains at most one of the a_j. If $q(a_j, A) < \varepsilon/8$ for some j, then this j is unique and we set $b = a_j, \rho = \varepsilon/4$. On the other hand, if $q(a_j, A) \geq \varepsilon/8$ for every j, then we set $b = A, \rho = \varepsilon/16$. In either case, none of the a_j lie in $\{w : 0 < q(w, b) \leq \rho\}$, and for large n the circle $C(0, r_n)$ lies in a component C_n of the set $\{z \in \mathbb{C} : q(f(z), b) < \rho\}$.

By Lemma 3.1, each C_n must be bounded, and so there must be infinitely many distinct such C_n. But, by Lemma 3.1 again, all but finitely many of the C_n are simply connected, and this is obviously a contradiction. This proves Theorem 1.3.

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