BANK-LAINE FUNCTIONS WITH SPARSE ZEROS

J.K. LANGLEY

Abstract. A Bank-Laine function is an entire function E satisfying $E'(z) = \pm 1$ at every zero of E. We construct a Bank-Laine function of finite order with arbitrarily sparse zero-sequence. On the other hand, we show that a real sequence of at most order 1, convergence class, cannot be the zero-sequence of a Bank-Laine function of finite order.

1. Introduction

A Bank-Laine function is an entire function E such that $E'(z) = \pm 1$ at every zero z of E. These arise from differential equations in the following way [1, 12].

Let A be an entire function, and let f_1, f_2 be linearly independent solutions of

$$w'' + A(z)w = 0, \tag{1}$$

normalized so that the Wronskian $W = W(f_1, f_2) = f_1 f_2' - f_1' f_2$ satisfies $W = 1$. Then $E = f_1 f_2$ satisfies

$$4A = (E'/E)^2 - 2E''/E - 1/E^2. \tag{2}$$

Further, E is a Bank-Laine function while, conversely, if E is any Bank-Laine function then [3] the function A defined by (2) is entire, and E is the product of linearly independent normalized solutions of (1).

Extensive work in recent years has concerned the exponent of convergence $\lambda(f_j)$ of the zeros of solutions f_j, in connection with the order of growth $\rho(A)$ of the coefficient A, these defined by

$$\lambda(f_j) = \limsup_{r\to\infty} \frac{\log^+ N(r, 1/f_j)}{\log r}, \quad \rho(A) = \limsup_{r\to\infty} \frac{\log^+ T(r, A)}{\log r}. \tag{3}$$

It has been conjectured that

$$A \quad \text{transcendental}, \quad \rho(A) < \infty, \quad \max\{\lambda(f_1), \lambda(f_2)\} < \infty \tag{4}$$

implies that $\rho(A)$ is a positive integer, and this has been proved in [1] under the stronger assumption $\max\{\lambda(f_1), \lambda(f_2)\} < \rho(A) < \infty$. Further, (4) implies that $\rho(A) > 1/2$ [16, 17] and that E has finite order [1]. We refer the reader to [5, 10, 12, 15] for further results.

It was observed by Shen [18] that if (a_n) is a complex sequence tending to infinity without repetition, then there exists a Bank-Laine function F with zero-sequence (a_n), the construction based on the Mittag-Leffler theorem. A natural question arising from both this observation and the conjecture above is the following: for which sequences (a_n) with finite exponent of convergence does there exist a Bank-Laine function E of finite order with zero-sequence (a_n)? In [6] the answer was

1991 *Mathematics Subject Classification.* 30D35.

shown to be negative for certain special sequences, such as $a_n = n^2$. The following theorem shows that the answer is negative for a large class of sequences.

Theorem 1.1. *Let L be a straight line in the complex plane and let (a_n) be a sequence of pairwise distinct complex numbers, all lying on L, such that $|a_n| \to \infty$ as $n \to \infty$ and*

$$\sum_{a_n \neq 0} |a_n|^{-1} < \infty. \tag{5}$$

Then there is no Bank-Laine function of finite order with zero-sequence (a_n).

Obvious examples such as $E(z) = \sin z$ show that the hypothesis (5) is not redundant in Theorem 1.1. We shall see in Theorem 1.3 below that the hypothesis that all a_n lie on a line cannot be deleted either.

One obvious way to make Bank-Laine functions of finite order is to choose A to be a polynomial in (1): if A is not identically zero and has degree n then $\rho(E) = (n+2)/2$ [1]. However, there are very few examples in the literature of Bank-Laine functions of finite order associated via (2) with transcendental coefficient functions A. The simplest [1, 14, 18] are of the following form: given any polynomial P having only simple zeros, there exists a non-constant polynomial Q such that Pe^Q is a Bank-Laine function. A second class arises from equations having periodic coefficients [2, 4], leading to Bank-Laine functions of form $E(z) = P(e^{\alpha z}) \exp(\beta z)$, with P a polynomial and α, β constants. In view of the conjecture above and non-existence results such as Theorem 1.1, it seems worth looking for further examples.

Theorem 1.2 ([14])**.** *There exists a Bank-Laine function $F(z)$ of finite order, with infinitely many zeros and with transcendental associated coefficient function A, but having no representation of the form $F(z) = P(e^{\alpha z}) \exp(Q(z))$, with P, Q polynomials and α constant.*

It is relatively straightforward to show that the examples F of Theorem 1.2 cannot have a representation $F(z) = P_1(z)P_2(e^{\alpha z})e^{Q(z)}$, with P_1, P_2, Q polynomials and α a non-zero constant. For if $P_2(\beta) = 0$ and $e^{\alpha z} = \beta$ then

$$P_1(z)^2 e^{2Q(z)} = (\alpha\beta)^{-2} P_2'(\beta)^{-2}$$

and $Q(z) + \log P_1(z)$ would be a polynomial, by Lemma 5 of [13]. However, the use of quasiconformal modifications in the proof of Theorem 1.2 makes it difficult to determine precisely the form of the examples F, although it is clear from the distortion theorems used there that the exponent of convergence of the zeros of F will always be positive. A natural question is then whether there exist Bank-Laine functions of finite order with zeros which are infinite in number but have zero exponent of convergence, and we give a strongly affirmative answer to this question.

Theorem 1.3. *Let (c_n) be a positive sequence tending to $+\infty$. Then there exists a Bank-Laine function*

$$E(z) = e^z \prod_{n=1}^{\infty} (1 - z/\alpha_n),$$

with $|\alpha_n| > c_n$ for each n. Further, $\rho(E) = 1$ and $\lambda(E) = 0$ and E is the product $f_1 f_2$ of normalized linearly independent solutions of an equation (1), with A transcendental, and f_1 has no zeros.

Thus there exist Bank-Laine functions of finite order with arbitrarily sparse zero-sequences. The proof of Theorem 1.3 is lengthy but elementary, and it will be seen in the proof that the α_n lie close to, but not on, the imaginary axis.

2. Proof of Theorem 1.1

We assume that (a_n) is as in the statement of Theorem 1.1, and that there exists a Bank-Laine function E of finite order, with zero-sequence (a_n). There is no loss of generality in assuming that L is the real axis and all the a_n are non-zero, and that infinitely many a_n are positive. By (5) and [9, Chapter 1] we may write

$$E(z) = e^{P(z)+iQ(z)} \prod_{n=1}^{\infty} (1 - z/a_n) = e^{P(z)+iQ(z)} W(z), \tag{6}$$

in which P and Q are polynomials, real on the real axis. Since the a_n are real and E is a Bank-Laine function, (6) implies that $e^{2iQ(a_n)}$ is real and positive and hence $e^{iQ(a_n)} = \pm 1$ for each n. Thus $E(z)e^{-iQ(z)}$ is a Bank-Laine function and there is no loss of generality in assuming that $Q(z) \equiv 0$.

Now E is the product $f_1 f_2$ of normalized linearly independent solutions of an equation (1), with A an entire function of finite order, and A and E are related by (2). By (2) and [9, Theorem 1.11, p.27], we have

$$T(r, A) = O(T(r, E)), \quad T(r, W) = o(r), \quad r \to \infty. \tag{7}$$

Lemma 2.1. *Let $\varepsilon > 0$ and let $z = re^{i\theta}$ with $r > 0$ and $\pm\theta \in (\varepsilon, \pi - \varepsilon)$. Then*

$$\log |W(z)| = o(r), \quad |W'(z)/W(z)| + |W''(z)/W(z)| = o(1), \quad r \to \infty. \tag{8}$$

Lemma 2.1 is an immediate consequence of the Poisson-Jensen formula [9, p.1] and its differentiated form [9, p.22], as well as of the fact that for z as in Lemma 2.1 the distance from z to the nearest zero of E is at least cr, in which the positive constant c depends only on ε.

Lemma 2.2. *P is not constant.*

Proof. Suppose that $P(z)$ is constant. Let y be real, with $|y|$ large. Then

$$2 \log |W(iy)| = \sum_{n=1}^{\infty} \log(1 + y^2/a_n^2) = \log M(y^2, G), \quad G(z) = \prod_{n=1}^{\infty} (1 + z/a_n^2), \tag{9}$$

and so $|W(iy)|$ is large, since G is a transcendental entire function in (9). Thus $A(iy) = o(1)$, using (2) and (8). A standard application of the Phragmén-Lindelöf principle now shows that either $A(z) \equiv 0$, which is obviously impossible, or A has at least order 1, mean type. However, (7) gives $T(r, A) = o(r)$, and this is a contradiction.

□

Thus P is a non-constant real polynomial. Now if $P(x)$ is negative for large positive x, we have $W'(x)e^{P(x)} \to 0$ as $x \to +\infty$, using (7), which contradicts our earlier assumption that E has infinitely many zeros on the positive real axis. There must therefore exist positive constants c_j such that

$$|\arg P(z)| < \pi/2 - c_1, \quad |z| > c_2, \quad |\arg z| < c_3. \tag{10}$$

Let δ be a small positive constant. Then (2), (8) and (10) give

$$A(z) = -\frac{1}{4}P'(z)^2(1+o(1)), \tag{11}$$

for $|z| > c_2, \delta < |\arg z| < c_3$. We now apply the Phragmén-Lindelöf principle to the function $A(z)P'(z)^{-2}$, which has finite order, and deduce that (11) holds for large z with $|\arg z| < c_3$.

The contradiction required to prove Theorem 1.1 arises at once upon applying the following lemma.

Lemma 2.3. *Let c be a positive constant. Then there exists a positive constant δ such that the following is true. Suppose that $A(z)$ is analytic and satisfies* (11) *as $z \to \infty$ in the region S given by $|z| \geq r_0, |\arg z| \leq \delta$, in which P is a polynomial of positive degree N satisfying $|\arg P(z)| < \pi/2 - 2c$ as $z \to \infty$ in S. Let f be a non-trivial solution of* (1) *in S. Then ff' has finitely many zeros in S.*

Proof. This is a standard application of Green's transform as in [11, pp.286-8]. Let ε be small and positive, and assume that ff' has infinitely many zeros in S. We may write

$$P(z) = bz^N(1+o(1)), \quad \arg P'(z) = (N-1)\arg z + \alpha + o(1), \quad \alpha = \arg b,$$

as $z \to \infty$. Thus, without loss of generality, we have

$$|\alpha| \leq \pi/2 - c, \quad 2c \leq \pi + 2\alpha \leq 2\pi - 2c. \tag{12}$$

Also, as $z \to \infty$ in S, provided δ was chosen small enough,

$$\pi + 2\alpha - \varepsilon \leq \arg A(z) \leq \pi + 2\alpha + \varepsilon. \tag{13}$$

Suppose now that z_0 and z_1 are zeros of ff' in S with $|z_0|$ and $|z_1/z_0|$ large. Following [11, pp.286-8], write

$$z = z_0 + re^{is}, \quad z_1 = z_0 + Re^{is}, \quad F(r) = f(z_0 + re^{is}), \quad H(r) = \overline{F(r)}F'(r)$$

with $r, R > 0$ and s real. Then

$$H'(r) = |F'(r)|^2 + \overline{F(r)}F''(r) = |F'(r)|^2 - e^{2is}A(z)|f(z)|^2$$

and hence

$$I = \int_0^R |F'(r)|^2 dr = \int_0^R e^{2is}A(z_0 + re^{is})|f(z_0 + re^{is})|^2 dr. \tag{14}$$

If z_1 is large enough then without loss of generality $|s| < 4\delta$ and hence, using (13),

$$\pi + 2\alpha - \varepsilon - 8\delta \leq \arg I \leq \pi + 2\alpha + \varepsilon + 8\delta.$$

On the other hand we obviously have $I > 0$, by (14). Provided ε and δ were chosen small enough we thus have $-c + 2k\pi < \pi + 2\alpha < c + 2k\pi$ for some integer k, which contradicts (12).

$\square$

From Lemma 2.3 we deduce the following result.

Theorem 2.1. *Let $E = We^P$ be a Bank-Laine function, with P a polynomial of positive degree N and W an entire function of order $\rho(W) < N$. Let $\theta_1 < \theta_2$ and $c > 0$ and suppose that $|\mathrm{Re}(P(z))| > c|z|^N$ as $z \to \infty$ in the sector S given by $\theta_1 \leq \arg z \leq \theta_2$. Then E has finitely many zeros in S.*

Thus zeros of E can only accumulate near the rays on which $\mathrm{Re}(P(z)) = o(|z|^N)$. A example illustrating this result is $E(z) = (1/\pi)\sin(\pi z)\exp(2\pi i z^2)$.

Proof. Obviously we have $|\mathrm{Re}(P(z))| > (c/2)|z|^N$ as $z \to \infty$ in a slightly larger sector S_1. Now suppose that $\theta_1 \leq \theta \leq \theta_2$ and that E has infinitely many zeros in every sector $|\arg z - \theta| < \delta, \delta > 0$. We may assume that $\theta = 0$.

Now if $\mathrm{Re}(P(z)) < -(c/2)|z|^N$ as $z \to \infty$ in S_1 then E and E' are small in S_1 and the result is obvious. Suppose now that $\mathrm{Re}(P(z)) > (c/2)|z|^N$ for large z in S_1. By (2) there exists an entire function A of finite order such that E is the product of linearly independent solutions of (1). Further, by standard estimates [8, 9] there is a set H_0 of measure 0 such that for all real θ not in H_0 we have, for $z = re^{i\theta}, r > 0$,

$$\log|W(z)| = o(r^N), \quad W'(z)/W(z) = o(r^{N-1}), \quad W''(z)/W(z) = o(r^{2N-2}).$$

Then we have (11) for large z in S_1 with $\arg z \notin H_0$ and hence, by the Phragmén-Lindelöf principle, for all large z in S. Applying Lemma 2.3 gives a contradiction, if δ is small enough. □

3. Proof of Theorem 1.3

Let λ be a large positive constant. There is no loss of generality in assuming that

$$c_1 > \lambda^2, \quad c_{j+1}/c_j > \lambda^2, \quad j = 1, 2, \ldots. \tag{15}$$

Choose $A_1, A_2, \ldots$ inductively, so that $|A_1| > \lambda c_1$ and $e^{A_1}(-1/A_1) = 1$, while

$$|A_j| > \lambda c_j, \quad |A_{j+1}/A_j| > \lambda^2, \tag{16}$$

and

$$e^{A_j}(-1/A_j) \prod_{1 \leq \mu < j} (1 - A_j/A_\mu) = 1 \tag{17}$$

for each j. To see that such A_j exist, we need only note that the left hand side of (17) is a meromorphic function of A_j with finitely many zeros and poles. Let

$$D_j = \{A_j + \alpha + i\beta : \quad -\pi \leq \alpha \leq \pi, \quad -\pi \leq \beta \leq \pi \quad \}. \tag{18}$$

Provided λ was chosen large enough we then have, by (16),

$$|a_j| > c_j, \quad |a_\mu/a_j| > \lambda^{\mu-j}, \quad a_j \in D_j, \quad a_\mu \in D_\mu, \quad \mu > j. \tag{19}$$

We also have

$$|a_\mu - a_j| \geq (1 - 1/\lambda)\max\{|a_j|, |a_\mu|\}, \quad a_j \in D_j, \quad a_\mu \in D_\mu, \quad j \neq \mu. \tag{20}$$

For positive integer n and $1 \leq j \leq n$ and a_j lying in an open neighbourhood of D_j, define

$$F_{j,n}(a_1, \ldots, a_n) = e^{a_j} G_{j,n}(a_1, \ldots, a_n) = e^{a_j}(-1/a_j) \prod_{1 \leq \mu \leq n, \mu \neq j} (1 - a_j/a_\mu). \tag{21}$$

For the proof of Theorem 1.3 we need a number of lemmas.

Lemma 3.1. *Suppose that $\delta > 0$ and that $a_j, b_j \in D_j$ and $|a_j - b_j| \leq \delta$ for $j = 1, \ldots, n$. Then, for $j = 1, \ldots, n$,*

$$\left|\log \frac{G_{j,n}(a_1, \ldots, a_n)}{G_{j,n}(b_1, \ldots, b_n)}\right| \leq \frac{6\delta}{\lambda(1 - 1/\lambda)^2}. \tag{22}$$

Proof. By (21) we may write

$$-G_{j,n}(a_1,\ldots,a_n)=\prod_{1\le\mu\le n}a_\mu^{-1}\prod_{1\le\mu\le n,\mu\ne j}(a_\mu-a_j). \tag{23}$$

Now, using (20),

$$\left|\frac{a_\mu-a_j}{b_\mu-b_j}-1\right|\le\frac{2\delta}{(1-1/\lambda)\max\{|b_\mu|,|b_j|\}}.$$

Using (19) and the fact that $|\log(1+z)|\le 2|z|$ for $|z|\le 1/2$, this gives

$$\left|\sum_{1\le\mu\le n,\mu\ne j}\log\frac{a_\mu-a_j}{b_\mu-b_j}\right|\le\frac{4\delta}{(1-1/\lambda)}\sum_{\mu=1}^n\frac{1}{|b_\mu|}\le\frac{4\delta}{\lambda(1-1/\lambda)^2}. \tag{24}$$

Similarly

$$\left|\sum_{1\le\mu\le n}\log\frac{b_\mu}{a_\mu}\right|\le 2\sum_{1\le\mu\le n}\frac{\delta}{|a_\mu|}\le\frac{2\delta}{\lambda(1-1/\lambda)}.$$

On combination with (24) this proves Lemma 3.1.

□

Lemma 3.2. *Let n be a positive integer and let $a_j\in D_j$ for $1\le j\le n$. Then the Jacobian matrix*

$$J=\left(\frac{\partial F_{j,n}}{\partial a_k}\right)$$

is non-singular.

Proof. It suffices to show that the Jacobian matrix

$$H=\left(\frac{\partial g_j}{\partial a_k}\right),\quad g_j=\log F_{j,n}\quad , \tag{25}$$

is non-singular, since the mapping $\phi(w_1,\ldots,w_n)=(e^{w_1},\ldots,e^{w_n})$ has non-singular Jacobian matrix. Now, by (21),

$$\frac{\partial g_j}{\partial a_j}=1-\frac{1}{a_j}+\sum_{1\le\mu\le n,\mu\ne j}\frac{1}{a_j-a_\mu}$$

and so, using (19) and (20), we have

$$\left|\frac{\partial g_j}{\partial a_j}-1\right|\le\frac{1}{|a_j|}+\frac{1}{(1-1/\lambda)}\sum_{1\le\mu\le n,\mu\ne j}\frac{1}{|a_\mu|}\le\frac{1}{\lambda(1-1/\lambda)^2}. \tag{26}$$

Further, for $k\ne j$, using (21),

$$\frac{\partial g_j}{\partial a_k}=\frac{a_j}{a_k(a_k-a_j)}$$

which gives, using (19) and (20) again,

$$\left|\frac{\partial g_j}{\partial a_k}\right|\le\frac{1}{(1-1/\lambda)|a_k|}\le\frac{1}{(1-1/\lambda)\lambda^k}. \tag{27}$$

Using (26) and (27) we may now write

$$H=I_n+C,\quad C=(c_{j,k}), \tag{28}$$

in which I_n is the n by n identity matrix and the entries $c_{j,k}$ of C satisfy

$$|c_{j,j}| \leq \frac{1}{\lambda(1-1/\lambda)^2}, \quad |c_{j,k}| \leq \frac{1}{(1-1/\lambda)\lambda^k}, \quad j \neq k. \tag{29}$$

Let d be a column vector with entries $d_1, \ldots, d_n$ and let d_r have greatest modulus, say σ. Then by (29), each entry of Cd has modulus at most

$$\sigma\left(\frac{1}{\lambda(1-1/\lambda)^2} + \frac{1}{(1-1/\lambda)}\sum_{k=1}^{n}\frac{1}{\lambda^k}\right) \leq \frac{2\sigma}{\lambda(1-1/\lambda)^2} < \sigma$$

provided λ was chosen large enough. Thus Hd cannot be the zero vector.

□

Lemma 3.3. *Suppose that $a_\mu \in D_\mu$ for $1 \leq \mu \leq n$ and that $a_j \in \partial D_j$ for some j with $1 \leq j \leq n$. Then*

$$|F_{j,n}(a_1, \ldots, a_n) - 1| \geq \frac{1}{4}. \tag{30}$$

Proof. By (17) and (21) we have

$$F_{j,n}(A_1, \ldots, A_n) = \prod_{j<\mu\leq n}(1 - A_j/A_\mu)$$

and so

$$|\log F_{j,n}(A_1, \ldots, A_n)| \leq 2\sum_{j<\mu\leq n}\left|\frac{A_j}{A_\mu}\right| \leq \frac{2}{\lambda-1}, \tag{31}$$

using (16). In particular, $F_{j,n}(A_1, \ldots, A_n)$ is close to 1, provided λ was chosen large enough. Also,

$$\frac{F_{j,n}(a_1, \ldots, a_n)}{F_{j,n}(A_1, \ldots, A_n)} = e^{a_j - A_j}X_j = e^{a_j - A_j}\frac{G_{j,n}(a_1, \ldots, a_n)}{G_{j,n}(A_1, \ldots, A_n)}. \tag{32}$$

Now if $\mathrm{Re}(w) = -\pi$ then $|e^w - 1| \geq 1 - e^{-\pi} \geq 1/2$ while if $\mathrm{Re}(w) = \pi$ then $|e^w - 1| \geq e^\pi - 1 \geq 1/2$. If $\mathrm{Im}(w) = \pm\pi$ then e^w is real and negative and $|e^w - 1| \geq 1$. Thus for $a_j \in \partial D_j$ we have $|e^{a_j - A_j} - 1| \geq 1/2$. But X_j is close to 1, by Lemma 3.1, provided λ was chosen large enough, and Lemma 3.3 now follows.

□

The next lemma is the key step in proving Theorem 1.3.

Lemma 3.4. *For each positive integer n there exist $a_{1,1}, \ldots, a_{n,n}$ with $a_{j,n} \in D_j$ and*

$$F_{j,n}(a_{1,n}, \ldots, a_{n,n}) = 1, \quad 1 \leq j \leq n.$$

Proof. We set $a_{1,1} = A_1$ and the result is trivially true for $n = 1$. Assume now that $b_j = a_{j,n}$ have been chosen so that

$$b_j \in D_j, \quad F_{j,n}(b_1, \ldots, b_n) = 1, \quad 1 \leq j \leq n. \tag{33}$$

Now for $1 \leq j \leq n$, by (21),

$$\begin{aligned} F_{j,n+1}(b_1, \ldots, b_n, A_{n+1}) &= e^{b_j}(-1/b_j)(1 - b_j/A_{n+1})\prod_{1\leq\mu\leq n, \mu\neq j}(1 - b_j/b_\mu) \\ &= F_{j,n}(b_1, \ldots, b_n)(1 - b_j/A_{n+1}) \end{aligned}$$

and so

$$|F_{j,n+1}(b_1, \ldots, b_n, A_{n+1}) - 1| = \left| \frac{b_j}{A_{n+1}} \right| \leq \lambda^{j-n-1}, \tag{34}$$

using (19) and (33). Also, by (17),

$$\begin{aligned} F_{n+1,n+1}(b_1, \ldots, b_n, A_{n+1}) &= \frac{F_{n+1,n+1}(b_1, \ldots, b_n, A_{n+1})}{F_{n+1,n+1}(A_1, \ldots, A_n, A_{n+1})} \\ &= \frac{G_{n+1,n+1}(b_1, \ldots, b_n, A_{n+1})}{G_{n+1,n+1}(A_1, \ldots, A_n, A_{n+1})} \end{aligned}$$

and applying Lemma 3.1 gives

$$|F_{n+1,n+1}(b_1, \ldots, b_n, A_{n+1}) - 1| \leq \frac{24\pi}{\lambda(1 - 1/\lambda)^2}. \tag{35}$$

For $a_j \in D_j, 1 \leq j \leq n+1$, set

$$h(a_1, \ldots, a_{n+1}) = \sum_{j=1}^{n+1} |F_{j,n+1}(a_1, \ldots, a_{n+1}) - 1|^2 . \tag{36}$$

Then by (34) and (35), provided λ was chosen large enough,

$$h(b_1, \ldots, b_n, A_{n+1}) \leq \frac{(24\pi)^2}{\lambda^2(1 - 1/\lambda)^4} + \sum_{j=1}^{n} \lambda^{2(j-n-1)} < \frac{1}{16}. \tag{37}$$

However, if $a_\mu \in D_\mu$ for $1 \leq \mu \leq n+1$ and at least one a_j lies on ∂D_j, then by Lemma 3.3 we have $h(a_1, \ldots, a_{n+1}) \geq 1/16$. Choose $d_j \in D_j$ such that

$$h(a_1, \ldots, a_{n+1}) \geq h(d_1, \ldots, d_{n+1}), \quad a_j \in D_j.$$

Then d_j is an interior point of D_j for each j and, at $(d_1, \ldots, d_{n+1})$,

$$0 = \sum_{j=1}^{n+1} \left(\overline{F_{j,n+1}} - 1\right) \left(\frac{\partial F_{j,n+1}}{\partial a_k} \right), \quad 1 \leq k \leq n+1,$$

so that by Lemma 3.2 we have $F_{j,n+1}(d_1, \ldots, d_{n+1}) = 1$ for $1 \leq j \leq n+1$.

□

To complete the proof of Theorem 1.3, set

$$E_n(z) = e^z q_n(z), \quad q_n(z) = \prod_{1 \leq \mu \leq n} (1 - z/a_{\mu,n}).$$

Then E_n has one zero $a_{j,n}$ in each D_j, for $1 \leq j \leq n$, and

$$E_n'(a_{j,n}) = F_{j,n}(a_{1,n}, \ldots, a_{n,n}) = 1,$$

by Lemma 3.4. Let r be large and positive, with $|A_N| \leq r < |A_{N+1}|$. Then for positive integer m and $|z| \leq r$ we have, using (19),

$$\begin{aligned} |q_m(z)| &\leq (1+r)^{N+1} \prod_{N+2 \leq j \leq m} (1 + r/|a_{j,m}|) \\ &\leq (1+r)^{d \log r} \prod_{p=1}^{\infty} (1 + \lambda^{-p}) \\ &\leq \exp(2d(\log r)^2), \end{aligned}$$

using d to denote a positive constant independent of r and m. It follows that a subsequence q_{n_k} converges locally uniformly in the plane to an entire function q of order 0, and $q(0) = 1$. Set $E(z) = e^z q(z)$. By the usual diagonalization process we may assume that

$$\lim_{k\to\infty} a_{j,n_k} = \alpha_j \in D_j$$

for each j. Thus $E(\alpha_j) = 0$ and $E'(\alpha_j) = 1$ for each j. Further, if $E(\alpha) = 0$ then by Hurwitz' theorem each q_{n_k}, for k large, has a zero near α. Thus the α_j are the only zeros of E and E has precisely one zero in each D_j.

It remains only to observe that the coefficient function A associated with E has order at most 1, by (2), and is transcendental, since $m(r, 1/E) \neq O(\log r)$, while f_1 has no zeros since $E'(\alpha_j) = 1$ and $W(f_1, f_2) = 1$. Theorem 1.3 is proved.

A natural question to ask is whether examples such as that above could be constructed more elegantly using techniques of interpolation theory [7]. However Theorem 1.1 makes it clear that one cannot arbitrarily specify the zero-sequence of a Bank-Laine function of finite order, and it seems necessary to allow the location of the zeros to vary as in Lemma 3.4 above.

References

[1] S. Bank and I. Laine, *On the oscillation theory of $f'' + Af = 0$ where A is entire*, Trans. Amer. Math. Soc. **273** (1982), 351-363.

[2] S. Bank and I. Laine, *Representations of solutions of periodic second order linear differential equations*, J. reine angew. Math. **344** (1983), 1-21.

[3] ———, *On the zeros of meromorphic solutions of second-order linear differential equations*, Comment. Math. Helv. **58** (1983), 656-677.

[4] S. Bank, I. Laine and J. K. Langley, *On the frequency of zeros of solutions of second order linear differential equations*, Results. Math. **10** (1986), 8-24.

[5] S. Bank and J. K. Langley, *On the oscillation of solutions of certain linear differential equations in the complex domain*, Proc. Edin. Math. Soc. **30** (1987), 455-469.

[6] S. M. ElZaidi, *On Bank-Laine sequences*, Complex Variables **38** (1999), 201-200.

[7] J. B. Garnett, *Bounded analytic functions*, Academic Press, New York 1981.

[8] G. Gundersen, *Estimates for the logarithmic derivative of a meromorphic function, plus similar estimates*, J. London Math. Soc. (2) **37** (1988), 88-104.

[9] W. K. Hayman, *Meromorphic functions*, Oxford at the Clarendon Press, 1964.

[10] S. Hellerstein, J. Miles and J. Rossi, *On the growth of solutions of certain linear differential equations*, Ann. Acad. Sci. Fenn. Ser. A. I. Math. **17** (1992), 343-365.

[11] E. Hille, *Ordinary differential equations in the complex domain*, Wiley, New York, 1976.

[12] I. Laine, *Nevanlinna theory and complex differential equations*, de Gruyter Studies in Math. **15**, Walter de Gruyter, Berlin/New York 1993.

[13] J. K. Langley, *On second order linear differential polynomials*, Results. Math. **26** (1994), 51-82.

[14] ———, *Quasiconformal modifications and Bank-Laine functions*, Arch. Math. **71** (1998), 233-239.

[15] J. Miles and J. Rossi, *Linear combinations of logarithmic derivatives of entire functions with applications to differential equations*, Pacific J. Math. **174** (1996), 195-214.

[16] J. Rossi, *Second order differential equations with transcendental coefficients*, Proc. Amer. Math. Soc. **97** (1986), 61-66.

[17] L. C. Shen, *Solution to a problem of S. Bank regarding the exponent of convergence of the solutions of a differential equation* $f'' + Af = 0$, Kexue Tongbao **30** (1985), 1581-1585.

[18] ———, *Construction of a differential equation $y'' + Ay = 0$ with solutions having prescribed zeros*, Proc. Amer. Math. Soc. **95** (1985), 544-546.

School of Mathematical Sciences, University of Nottingham, NG7 2RD UK
E-mail address: jkl@maths.nott.ac.uk