BANK-LAINE FUNCTIONS VIA QUASICONFORMAL SURGERY

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ABSTRACT. Using quasiconformal surgery we construct new examples of entire functions E such that $E(z) = 0$ implies $E'(z) = \pm 1$, these associated with second order linear differential equations with transcendental coefficients. We also extend some previous results on the zero sequences of such functions.

1. INTRODUCTION

A Bank-Laine function [20, 21] is an entire function E such that $E'(z) = \pm 1$ whenever $E(z) = 0$. These functions arise from differential equations [2]. Indeed, let A be entire, and f_1, f_2 be linearly independent solutions of the equation

$$w'' + A(z)w = 0, \tag{1}$$

normalized so that the Wronskian $W = W(f_1, f_2) = f_1 f_2' - f_1' f_2$ satisfies $W = 1$. Then $E = f_1 f_2$ is a Bank-Laine function and satisfies

$$4A = (E'/E)^2 - 2E''/E - 1/E^2. \tag{2}$$

Conversely, if E is any Bank-Laine function then [4] the function A defined by (2) is entire, and E is the product of linearly independent normalized solutions of (1) [2]. Extensive research in recent years has concerned the exponent of convergence $\lambda(f_j)$ of the zeros of solutions f_j, in connection with the order of growth $\rho(A)$ and lower order $\mu(A)$ of the coefficient A, these defined using standard notation from [14] by

$$\begin{aligned}
\rho(A) &= \limsup_{r\to\infty} \frac{\log^+ T(r,A)}{\log r}, \\
\mu(A) &= \liminf_{r\to\infty} \frac{\log^+ T(r,A)}{\log r}, \\
\lambda(f_j) &= \limsup_{r\to\infty} \frac{\log^+ N(r,1/f_j)}{\log r}. \qquad (3)
\end{aligned}$$

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The *Bank-Laine conjecture* asserts that the condition

$$A \quad \text{transcendental}, \quad \rho(A) < \infty, \quad \max\{\lambda(f_1), \lambda(f_2)\} < \infty \tag{4}$$

implies that $\rho(A)$ is a positive integer, and this is known to be true [2] under the stronger condition $\max\{\lambda(f_1), \lambda(f_2)\} < \rho(A) < \infty$. It is known further that (4) implies that $\rho(A) > 1/2$ [26, 27] and that E has finite order [2]. Further results, and analogues for higher order equations, may be found in [2, 3, 4, 5, 6, 7, 15, 16, 17, 18, 19, 24].

Note that if E is a Bank-Laine function of finite order then by (2) the associated coefficient function A satisfies

$$T(r, A) = m(r, A) = 2m(r, 1/E) + O(\log r). \tag{5}$$

In particular, if $\delta(0, E) > 0$ then A is transcendental.

It appears to be relatively difficult to construct Bank-Laine functions of finite order associated via (2) with transcendental coefficient functions A. The simplest [2, 20, 28] are of the following form: given any polynomial P having only simple zeros, Lagrange interpolation gives a non-constant polynomial Q such that Pe^Q is a Bank-Laine function. A second class arises from equations having periodic coefficients [3], leading to Bank-Laine functions of form $E(z) = P(e^{\alpha z}) \exp(\beta z)$, with P a polynomial and α, β constants. Of course, all examples arising in this way have $\lambda(E) \leq 1$. We also note that in [20] quasiconformal modifications, applied to the quotient f_1/f_2, were used to convert a Bank-Laine function $f_1(z)f_2(z) = \exp(q(z)) \sin \pi z$, with q a polynomial, into a Bank-Laine function $E(z)$ of finite order, with infinitely many zeros and transcendental associated coefficient function A, but having no representation of the form $E(z) = P(e^{\alpha z}) \exp(Q(z))$, with P, Q polynomials and α constant. This construction may be modified to give $\rho(E)$ finite but arbitrarily large, but we still do not know whether there are functions so obtained with $\infty > \lambda(E) > 1$.

Another result [21] shows that the zero set of a Bank-Laine function may be very sparse. Indeed, if (c_n) is a positive sequence tending to $+\infty$ then there exists a Bank-Laine function

$$E(z) = e^z \prod_{n=1}^{\infty} (1 - z/\alpha_n),$$

with $|\alpha_n| > c_n$ for each n. Further, $\rho(E) = 1$ and $\lambda(E) = 0$ and E is the product $f_1 f_2$ of normalized linearly independent solutions of an equation (1), with A transcendental, and f_1 has no zeros. These α_n may be chosen close to, but not on, the positive imaginary axis.

In the present paper we employ the quasiconformal surgery method of Shishikura [29] to produce examples with $\lambda(E) = \rho(E)$ finite but arbitrarily large.

Theorem 1.1. *Let n be a positive integer. Then there exists a Bank-Laine function E, associated with a transcendental coefficient function A, such* *that, in the notation* (1)*, the order* $\rho(E)$*, lower order* $\mu(E)$*, and exponent of convergence* $\lambda(E)$ *of the zeros of* E*, satisfy* $n \leq \mu(E) \leq \rho(E) = \lambda(E) < \infty$*.*

The construction of Theorem 1.1 starts from a function f of integer order $n \geq 2$ which satisfies the Bank-Laine condition at all but one of its zeros. Quasiconformal surgery is applied to $z + f(z)$, leading to a Bank-Laine function E which has $\lambda(E) = \rho(E) \geq \mu(E) \geq n$. It is not clear from our methods whether, as seems likely, the order $\rho(E)$ is preserved by this construction (and so would also be n). This is related to the question of how order is transformed by quasiconformal surgery.

We turn next to some results on the zero sequences of Bank-Laine functions. It was observed by Shen [28] that if (a_n) is a complex sequence tending to infinity without repetition, then there exists a Bank-Laine function F with zero sequence (a_n), the construction based on the Mittag-Leffler theorem. A natural question arising from both this observation and the Bank-Laine conjecture is the following: for which sequences (a_n) with finite exponent of convergence does there exist a Bank-Laine function E of finite order with zero sequence (a_n)? A negative answer for sufficiently sparse sequences lying on a line was given in [21]: if L is a straight line in the complex plane and (a_n) is a sequence of pairwise distinct complex numbers, all lying on L, such that $|a_n| \to \infty$ as $n \to \infty$ and

$$\sum_{a_n \neq 0} |a_n|^{-1} < \infty, \tag{6}$$

then there cannot exist a Bank-Laine function of finite order with zero sequence (a_n).

The obvious example $\sin z$ shows that the condition (6) is not redundant, but we show here that it may be weakened to minimal type, and to mean type if the a_n all lie on a ray.

Theorem 1.2. *Let* (a_n) *be a sequence of pairwise distinct real numbers, such that* $\lim_{n\to\infty} |a_n| = \infty$*, and let* $n(r)$ *be the number of* a_n *lying in* $|z| \leq r$*. Assume that at least one of the following holds:*
(i) we have $n(r) = o(r)$ *as* $r \to \infty$*;*
(ii) the a_n *are all positive, and* $n(r) = O(r)$ *as* $r \to \infty$*.*
Then there is no Bank-Laine function of finite order with zero sequence (a_n)*.*

It seems likely that if (a_n) is any strictly increasing positive sequence with limit ∞ then there is no Bank-Laine function of finite order with zero sequence (a_n), but the present methods do not suffice to prove this.

2. Lemmas needed for Theorem 1.1

For $a \in \mathbb{C}$ and $r > 0$ we use the standard notation

$$B(a, r) = \{z \in \mathbb{C} : |z - a| < r\}, \quad C(a, r) = \{z \in \mathbb{C} : |z - a| = r\}.$$

Lemma 2.1. *Let ϕ be a K-quasiconformal homeomorphism of the extended plane, fixing 0 and ∞. Then there exist positive constants d_j with the following properties.*
(i) $|z|^{1/d_1} \leq |\phi(z)| \leq |z|^{d_1}$ as $z \to \infty$;
(ii) $|\phi(z) - \phi(w)| \leq |z - w|^{d_2}|w|^{d_3}$ for large z, w with $|z - w| \leq 1$.
(iii) $|\phi(u)| \geq d_4|\phi(v)|$ for large u, v with $|u| = |v|$.

All assertions of Lemma 2.1 are standard [1, 22]. In particular, (*iii*) may be proved by considering $\psi(z) = \phi(zv)/\phi(v)$, since the K-quasiconformal mappings fixing $0, 1, \infty$ form an equicontinuous family.

The key tool for our construction is Shishikura's main lemma on quasiconformal surgery [8, 29], in the following form.

Lemma 2.2. *Let $g : \mathbb{C} \to \mathbb{C}$ be quasiregular, and let W be a non-empty open subset of $\mathbb{C}$. Assume that g maps W into W, and that $\overline{\partial} g = 0$ a.e. on W and on $\mathbb{C} \setminus g^{-N}(W)$, for some integer $N > 0$. Then there exist an entire function h and a quasiconformal homeomorphism ϕ of the extended plane fixing $0, 1, \infty$, such that $g \equiv \phi^{-1} \circ h \circ \phi$. Further, ϕ is conformal on W and on the interior of $\mathbb{C} \setminus \bigcup_{m=1}^{\infty} g^{-m}(W)$.*

Proof. We sketch the proof from [8, 29]. Set

$$W_0 = W, \quad W_{m+1} = g^{-m-1}(W) \setminus g^{-m}(W) \quad (m = 0, 1, 2, \ldots),$$

and

$$K = \mathbb{C} \setminus \bigcup_{m=0}^{\infty} g^{-m}(W).$$

Define a Beltrami coefficient $\mu(z)$ on $\mathbb{C}$ as follows. For $z \in W_0 \cup K$ we set $\mu = 0$. Assuming that μ has been defined on W_m, we then define μ for $w \in W_{m+1}$ by

$$\mu(w) = \frac{\mu_g(w) + \mu(g(w))A(w)}{1 + \mu(g(w))\overline{\mu_g(w)}A(w)}, \quad A = \frac{\overline{g_w}}{g_w}. \tag{7}$$

In this way, μ is defined inductively a.e. on $\mathbb{C}$ and, since $\mu_g(w) = 0$ a.e. on W_m for $m > N$, we have $|\mu(w)| \leq \kappa < 1$ a.e. We then define ϕ to be the quasiconformal homeomorphism of the extended plane fixing $0, 1, \infty$, and with complex dilatation μ, and (7) gives $\mu_{\phi \circ g} = \mu_\phi$ a.e., so that $\phi \circ g = h \circ \phi$, with h entire. $\square$

3. Proof of Theorem 1.1

Theorem 1.1 is trivially true when $n = 1$: we may take, for example, the function $E(z) = e^{2z} - e^z$; since $\delta(0, E) = \frac{1}{2}$, we see from (5) that the coefficient function A must be transcendental.

Now let $n \geq 2$ be an integer, and set

$$F(z) = z + f(z), \quad f(z) = \frac{e^{-2z^n} - e^{-z^n}}{nz^{n-1}}. \tag{8}$$

Thus f and F are entire, and f has zeros wherever $e^{z^n} = 1$. At such a zero z with $z \neq 0$ we have $f'(z) = -1$. Near 0 we have

$$f(z) \sim \frac{-2z^n + z^n}{nz^{n-1}} = -\frac{z}{n}.$$

It follows that all nonzero fixpoints of F are superattracting, i.e. have multiplier $F'(z) = 0$, while 0 is attracting with $F'(0) = 1 - 1/n \in (0,1)$. The new idea is to perform quasiconformal surgery so that this fixpoint is also superattracting.

Our first lemma of this section follows at once from (8).

Lemma 3.1. *For small positive δ and $j = 0, 1, \ldots, 2n-1$ let*

$$\theta_j = \frac{j\pi}{n}, \quad S_{j,\delta} = \{z : z \neq 0, |\arg z - \theta_j| < \frac{\pi}{2n} - \delta\}. \tag{9}$$

Then there exists $c(\delta) > 0$ such that, for large z in $S_{j,\delta}$,

$$\log|f(z)| > c(\delta)|z|^n \quad (j \quad \text{odd}), \quad \log|f(z)| < -c(\delta)|z|^n \quad (j \quad \text{even}).$$

Lemma 3.2. *Let λ be an unbounded path in the complex plane. Then F is unbounded on λ.*

Proof. We use a standard argument based on harmonic measure. Choose small positive constants δ and ε. As $z \to \infty$ in $S_{j,\delta}$ we have $F(z)/z \sim 1$ if j is even, and $F(z)/z \to \infty$ if j is odd.

Assume now that λ is an unbounded path on which $F(z)$ is bounded. Then $p(z) = F(z)/z$ is small for large z on λ. Thus there exists, for arbitrarily large positive R, a simple arc $\lambda_R \subseteq \lambda$ joining $C(0, \sqrt{R})$ to $C(0, R^2)$ and, apart from its endpoints, lying in $\sqrt{R} < |z| < R^2$, and with $|p(z)| < \varepsilon$ on λ_R. It is clear from Lemma 3.1 that λ_R must lie in one of the regions between the $S_{j,\delta}$. Since δ is small we may therefore connect $C(0, \sqrt{R})$ to $C(0, R^2)$ by a radial line segment τ_R lying in one of the sets $S_{j,2\delta} \setminus S_{j,4\delta}$ with j even, and such that the angular distance between τ_R and λ_R is at least δ but at most 5δ. Since j is even and R is large we have $|p(z) - 1| < \varepsilon$ on τ_R.

In this way we obtain a domain D_R bounded by λ_R, τ_R and small arcs $U_{\sqrt{R}}$ and V_{R^2} of $C(0, \sqrt{R})$ and $C(0, R^2)$. Since δ is small standard estimates for harmonic measure [31, p.117] now give that

$$\omega(z, D_R, U_{\sqrt{R}} \cup V_{R^2}) < R^{-4n}, \quad z \in D_R, \quad |z| = R,$$

so that since p has order at most n we obtain, provided R is large enough,

$$|p(z)(p(z) - 1)| < 2\varepsilon, \quad z \in D_R, \quad |z| = R.$$

Choosing $z \in D_R$ with $|z| = R$ and $|p(z)| = |p(z) - 1|$ gives an immediate contradiction. □

We apply next some completely standard facts from iteration theory [8, 9, 30]. The attracting fixpoint 0 of F lies in a simply connected component D of the Fatou set of F, with $F(D) \subseteq D$. Let H map D conformally onto $\Delta = B(0,1)$, with $H(0) = 0$ and $H'(0) > 0$, and define $G = H \circ F \circ H^{-1}$ on

Δ. Thus G maps Δ into itself, with $G(0) = 0$ and $G'(0) = 1 - 1/n$. Further, F has at least one critical point z_1 in D. Were this not the case F^{-1} would admit analytic continuation throughout D, since F has no finite asymptotic values and the boundary of D lies in the Julia set of F, and this would imply that F is a conformal self-map of D and G is a conformal self-map of Δ, contradicting the fact that $|G'(0)| < 1$.

The next step of our construction is based on [10, p.106]. Let $w_1 = H(z_1)$ and choose R such that $|w_1| < R < 1$ and such that G has no critical values on $|u| = R$.

Lemma 3.3. *Let V_1 be that component of the set $\{w : |G(w)| < R\}$ which contains 0. Let $U_1 = H^{-1}(V_1)$ and $U_2 = H^{-1}(B(0, R))$, and set $\gamma_j = \partial U_j$. Then the γ_j are disjoint analytic Jordan curves surrounding the origin, such that γ_2 separates γ_1 from 0. Further, F maps U_1 properly onto U_2, and γ_1 onto γ_2, the mapping q-to-one for some $q > 1$.*

Proof. The component V_1 is simply connected by the maximum principle, and therefore so is U_1. Next, $F(U_1)$ is bounded, being a subset of $H^{-1}(B(0, R))$, and so it follows from Lemma 3.2 that U_1 is bounded. Thus the closure of V_1 lies in Δ, and so G maps V_1 properly onto $B(0, R)$ and F maps U_1 properly onto U_2. By Schwarz' lemma V_1 contains a disc $B(0, R'), R' > R$. Thus G has at least one critical point in V_1, and the mapping is q-to-one for some $q > 1$. □

Lemma 3.4. *Let $P(z) = H(z)^q$ on $U_2 \cup \gamma_2$. Then $P(z)$ extends to a function continuous on $U_1 \cup \gamma_1$, quasiregular on U_1, with $P(z) \equiv H(F(z))$ on γ_1. Further, P maps U_1 into $B(0, R)$.*

Proof. Let Ω be the doubly connected domain bounded by γ_1 and γ_2, and let $z = s(\zeta)$ map an annulus A given by $S < |\zeta| < T$ conformally onto Ω, with inner boundary mapped to inner boundary. Since $|H(s(\zeta))| \to R$ as $|\zeta| \to S$ it follows from the reflection principle that $(H \circ s)^q$ has an analytic and univalent extension to a neighbourhood of each point of $C(0, S)$, and so has s. Similarly, since $H \circ F$ maps γ_1 onto $C(0, R)$, we see that $H \circ F \circ s$ extends analytically and univalently to a neighbourhood of each point of $C(0, T)$, and again so does s. For real t we may then write

$$P(s(Se^{it})) = R_1 \exp(i\psi_1(t)), \quad H(F(s(Te^{it}))) = R_2 \exp(i\psi_2(t)), \tag{10}$$

in each which $\psi_j : [-\pi, \pi] \to \mathbb{R}$ is C^1 with positive derivative, and

$$R_1 = R^q < R = R_2, \quad \psi_j(\pi) = \psi_j(-\pi) + 2\pi q. \tag{11}$$

Using (10) and (11) we readily extend P to the annulus $A = s^{-1}(\Omega)$ as $P(s(\zeta)) = \exp(Q(\zeta))$, in which, for $S \leq r \leq T$ and $-\pi \leq t \leq \pi$,

$$\left(\log \frac{T}{S}\right) Q(re^{it}) = \left(\log \frac{r}{S}\right) (\log R_2 + i\psi_2(t)) - \left(\log \frac{r}{T}\right) (\log R_1 + i\psi_1(t)) . \tag{12}$$

A straightforward computation from (12) shows that $Q(\zeta)$ is a locally quasiconformal function of $\log \zeta$, its dilatation uniformly bounded since the ψ_j are C^1. Since $|P(z)| = R$ for $z \in \gamma_1$, the last assertion follows from the maximum principle. Lemma 3.4 is proved. $\square$

Lemma 3.5. *Define $g(z)$ on $\mathbb{C}$ by $g(z) = H^{-1}(P(z))$ for $z \in U_1$, with P as in Lemma 3.4, and set $g(z) = F(z)$ for $z \in \mathbb{C} \setminus U_1$. Then:*
(i) g is analytic on $W = U_2$ and on the exterior of γ_1;
(ii) g is quasiregular on $\mathbb{C}$;
(iii) g maps U_1 into U_2, and the closure of U_1 into U_1;
(iv) $\overline{\partial} g = 0$ on $\mathbb{C} \setminus g^{-2}(W)$;
(v) g has a fixpoint at 0, and no other fixpoint in the closure of U_1;
(vi) if z^* is a fixpoint of g then g is analytic near z^* and z^* is superattracting.

Proof. We first note that (i) is obvious. To prove (ii) we need only note that g is quasiregular on U_1 and analytic outside γ_1, continuous on $\mathbb{C}$, and that γ_1 has two-dimensional Lebesgue measure 0. Next, (iii) holds since P maps U_1 into $B(0, R)$, and (iv) follows, since if $g(g(z_0)) \notin W$ then z_0 lies outside γ_1 and $g(z) = F(z)$ near z_0. Obviously $g(0) = 0$, since $H(0) = 0$, and for z in the closure of U_1 the iterates g_m of g satisfy, using Lemma 3.4,

$$g_2(z) \in U_2, \quad g_{m+2}(z) = H^{-1}(H(g_2(z))^{q^m}) \to H^{-1}(0) = 0,$$

and this proves (v). Assertion (vi) is now obvious. $\square$

In particular, the fixpoint 0, which was only attracting for F in (8), is now superattracting for g. By Lemma 2.2, there exists a quasiconformal homeomorphism ϕ of the extended plane, fixing $0, 1, \infty$, and an entire function h with $g \equiv \phi^{-1} \circ h \circ \phi$. Further, ϕ is conformal on $W = U_2$ and on the interior of $\mathbb{C} \setminus \bigcup_{m=1}^{\infty} g^{-m}(W)$.

Lemma 3.6. *All fixpoints of h are superattracting, and z^* is a fixpoint of g if and only if $\phi(z^*)$ is a fixpoint of h.*

Proof. The second assertion of the lemma is obvious, and it is clear that 0 is a superattracting fixpoint of h, since ϕ is conformal near 0. Next, let $z^* \neq 0$ be a fixpoint of g. By Lemma 3.5, z^* is superattracting and there exists an invariant neighbourhood U^* of z^*, so that in particular $U^* \cap g^{-m}(W) = \emptyset$ for every $m \geq 1$. Thus ϕ is conformal near z^* and so $h'(\phi(z^*)) = 0$. $\square$

Lemma 3.7. *Let $E(z) = h(z) - z$. Then E is a Bank-Laine function of finite order.*

Proof. E is a Bank-Laine function since all fixpoints of h are superattracting, and E has finite order by Lemma 2.1(i). $\square$

Lemma 3.8. *There exist $c_0 > 0$ and a family of discs B_k, having finite sum of radii, such that*

$$|E''(z)/E(z)| + |E'(z)/E(z)| \leq |z|^{c_0}, \quad z \notin B = \bigcup_{k=1}^{\infty} B_k. \tag{13}$$

Further, each image $\psi(B_k)$ under the inverse map ψ of ϕ is contained in a disc B_k^*, these discs also having finite sum of radii.

Proof. Since E has finite order we may take $B_k = B(u_k, |u_k|^{-M_1})$ for $k \geq 2$, with M_1 a large positive constant and the u_k zeros of E, and (13) follows from a standard application of the differentiated Poisson-Jensen formula [14, p.22]. The assertion concerning the images $\psi(B_k)$ follows from Lemma 2.1 (i) and (ii). $\square$

Lemma 3.9. *Let δ be small and positive, and let θ_j and $S_{j,\delta}$, for $j = 0, 1, \ldots, 2n-1$, be as in* (9). *Then there exists $c > 0$ such that the following hold.*
(i) If j is odd then

$$\log|E(z)| \geq |z|^c, \quad z \to \infty, \quad z \in \phi(S_{j,\delta}). \tag{14}$$

(ii) If j is even then

$$\log|E(z)| \leq -|z|^c, \quad z \to \infty, \quad z \in \phi(S_{j,\delta}). \tag{15}$$

Further, each $\phi(S_{j,\delta})$ contains a path σ_j tending to infinity and meeting none of the discs B_k of Lemma 3.8*, on which*

$$(-1)^{j+1}\frac{\log|E(z)|}{\log|z|} \to +\infty \tag{16}$$

as $z \to \infty$.

Proof. We use $c_1, c_2, \ldots$ to denote positive constants. If j is odd then Lemma 3.1 implies that $g(\psi(z)) = F(\psi(z))$ is large for large z in $\phi(S_{j,\delta})$ so that using Lemma 2.1 (i) we get

$$\log|h(z)| = \log|\phi(g(\psi(z)))| \geq c_1 \log|F(\psi(z))| \geq c_2|\psi(z)|^n \geq c_3|z|^{n/c_4},$$

which gives (14). Next, let j be even. Then $F(\psi(z)) \sim \psi(z)$ for large z in $\phi(S_{j,\delta})$ and applying Lemma 2.1 (ii) and Lemma 3.1 we get

$$|g(\psi(z)) - \psi(z)| = |f(\psi(z))| < |\exp(-c(\delta)|\psi(z)|^n) < \exp(-c(\delta)|z|^{n/c_5}),$$

and so

$$|E(z)| = |h(z) - z| \leq |\psi(z)|^{c_6}|g(\psi(z)) - \psi(z)|^{c_7} < \exp(-c_8|z|^{n/c_9}).$$

Finally, the existence of a suitable path σ_j follows from Lemma 3.8: we need only ensure that $\psi(\sigma_j)$ lies in $S_{j,\delta}$ and avoids the discs B_k^*. $\square$

Since E is a Bank-Laine function, E is the product of linearly independent solutions of (1), in which A is given by (2) and is entire, with $\rho(A) \leq \rho(E)$.

Lemma 3.10. *The coefficient function A is transcendental, and the lower order τ of E is at least n.*

Proof. Take one of the paths σ_j of Lemma 3.9, with j even. Then E is small and A is large as z tends to infinity on σ_j, by (2), (13) and (16). This proves that A is transcendental.

We prove next that E has lower order $\tau \geq n$, the method being essentially that of [26]. By (2) and (13) there exists a positive integer N_1 such that for large z not in the exceptional set B of Lemma 3.8 we have

(17)
$$|E(z)| \geq |z|^{N_1} \Rightarrow |A(z)| \leq |z|^{N_1 - 1}, \quad |E(z)| \leq |z|^{-N_1} \Rightarrow |A(z)| \geq |z|^{N_1+1}.$$

Choose polynomials P_1, Q_1 of degree less than N_1 such that

$$E_1(z) = z^{-N_1}(E(z) - P_1(z)), \quad A_1(z) = z^{-N_1}(A(z) - Q_1(z))$$

are entire. Lemma 3.9 gives us $2n$ paths σ_j each tending to infinity and avoiding the exceptional set B, and such that $E_1(z) \to 0$ and $A_1(z) \to \infty$ on σ_j, if j is even, and vice versa if j is odd. With N_2 a large positive constant the set $\{z \in \mathbb{C} : |E_1(z)| > N_2\}$ therefore has at least n components $D_1, \ldots, D_n$, and similarly the set $\{z \in \mathbb{C} : |A_1(z)| > N_2\}$ has at least n components $D_{n+1}, \ldots, D_{2n}$. Further, the intersection $D_j \cap D_{j'}, j \neq j'$, lies in the exceptional set B of (13).

Let $\theta_j(t)$ denote the angular measure of the intersection of D_j with the circle $|z| = t > 0$. Since (5) gives

$$\log M(2r, A_1) \leq 3T(4r, A) + O(\log r) \leq 6T(4r, E) + O(\log r),$$

there is a sequence $r_n \to \infty$ with

$$\log M(2r_n, A_1) + \log M(2r_n, E_1) = O(r_n^{\tau + o(1)}), \quad n \to \infty.$$

Applying a standard estimate for harmonic measure [31, p.117] we get

(18)
$$\pi \int_1^{r_n} \frac{dt}{t\theta_j(t)} \leq (\tau + o(1)) \log r_n, \quad n \to \infty.$$

But a standard application of the Cauchy-Schwarz inequality gives

$$4n^2 \leq \sum_{j=1}^{2n} \theta_j(t) \sum_{j=1}^{2n} \frac{1}{\theta_j(t)} \leq (2\pi + o(1)) \sum_{j=1}^{2n} \frac{1}{\theta_j(t)},$$

which with (18) gives $n \leq \tau$.

□

Lemma 3.11. *There exists $K > 0$ such that $\log M(r, E) \leq Kn(Kr, 1/E)$ as $r \to \infty$. In particular, $\lambda(E) = \rho(E)$.*

Proof. Let

$$M(r, \phi) = \max\{|\phi(z)| : |z| = r\}, \quad m_0(r, \phi) = \min\{|\phi(z)| : |z| = r\}.$$

We have, using Lemma 2.1 (i),

(19)
$$\log M(m_0(r, \phi), h) \leq \log M(r, h \circ \phi) = \log M(\phi \circ F) \leq cr^n$$

as $r \to \infty$. On the other hand, since $\phi(z^*)$ is a zero of E for every zero z^* of f with z^* large, we get

$$cr^n \leq n(r, 1/f) \leq n(M(r, \phi), 1/E) + c \leq n(cm_0(r, \phi), 1/E) + c,$$

using (8) and Lemma 2.1 (*iii*), which with (19) proves the lemma. □

This completes the proof of Theorem 1.1.

4. A result needed for Theorem 1.2

Theorem 1.2 will be deduced from the following result.

Theorem 4.1. *Let E be a Bank-Laine function of finite order such that E has infinitely many zeros, all real, and such that E is real on the real axis. Then $\delta(0, E) < 1$.*

The examples e^z and $(1/\pi)\exp(2\pi i z^2)\sin(\pi z)$ show that the assumptions that E is real and has infinitely many zeros are not redundant in Theorem 4.1.

To prove Theorem 4.1, we assume that E satisfies the hypotheses, but that $\delta(0, E) = 1$. We assume further, without loss of generality, that E has infinitely many zeros on the positive real axis. By a theorem of Pfluger [25], ρ is a positive integer. Results from [11, 12, 13] give continuous functions $L(r), L_2(r)$ such that

$$L(r) > 0, \quad L(cr) = L(r)(1 + o(1)), \quad L_2(cr) = L_2(r) + o(1) \tag{20}$$

as $r \to \infty$, uniformly for $1 \leq c \leq 2$, and such that

$$\log |E(re^{i\theta})| = L(r)r^\rho(\cos \rho(\theta - L_2(r))) + o(1)) \tag{21}$$

uniformly in θ, as $re^{i\theta}$ tends to infinity outside a C^0 set [23], a union U of open discs $B(z_k, r_k)$ of centre z_k and radius r_k such that $\sum_{|z_k|<r} r_k = o(r)$ as $r \to \infty$.

Lemma 4.1. *We may take $L_2(r) \equiv 0$.*

Proof. We may certainly take $L_2(r) \equiv 0$ or $L_2(r) \equiv \pi/\rho$. To see this, just choose a small positive θ_0 such that the rays $\arg z = \pm\theta_0$ have bounded intersection with the C^0 set U. Since E is real on the real axis, we may apply (21) to each of these rays and obtain a contradiction unless $\sin \rho L_2(r) = o(1)$.

If $L_2(r) \equiv \pi/\rho$ then a standard application of (21) and the Phragmén-Lindelöf principle shows that $E(z)$ is small for large z with $|\arg z| < \pi/8\rho$, and so is $E'(z)$. But this contradicts the assumed existence of infinitely many zeros ζ of E on the positive real axis, at which $E'(\zeta) = \pm 1$. □

Lemma 4.2. *Let ε be small and positive. Then*

$$\begin{aligned} \frac{1}{E(z)} &= o(1), \\ \frac{E'(z)}{E(z)} &= \rho L(r) z^{\rho-1}(1+o(1)), \\ \left(\frac{E'}{E}\right)'(z) &= \rho(\rho-1)L(r)z^{\rho-2} + o(L(r)r^{\rho-2}), \end{aligned} \tag{22}$$

as $|z| = r \to \infty$, uniformly for $\varepsilon \le \arg z \le 2\varepsilon$.

Proof. Let r be large. Since all zeros of E are real, (20) and (21) and Lemma 4.1 give

$$\log|E(z)| = \mathrm{Re}(L(r)z^\rho) + o(L(r)r^\rho), \quad r/4 \le |z| \le 8r, \quad \varepsilon/4 \le \arg z \le 8\varepsilon. \tag{23}$$

Set

$$H(z) = \log E(z) - L(r)z^\rho, \quad h(z) = L(r)^{-1}r^{-\rho}H(z).$$

Then (23) and an application of the Borel-Carathéodory inequality give a constant d_r such that

$$h(z) = d_r + o(1), \quad r/2 \le |z| \le 4r, \quad \varepsilon/2 \le \arg z \le 4\varepsilon.$$

Thus

$$h'(z) = o(r^{-1}), \quad h''(z) = o(r^{-2}), \quad r \le |z| \le 2r, \quad \varepsilon \le \arg z \le 2\varepsilon,$$

and this gives (22). □

We also obtain from (2) that

$$4A = -(E'/E)^2 - 2(E'/E)' - 1/E^2$$

and (20) and (22) now give, using the fact that E and A are real on the real axis,

$$4A(z) = -\rho^2 L(r)^2 z^{2\rho-2}(1+o(1)), \quad \varepsilon \le |\arg z| \le 2\varepsilon. \tag{24}$$

Lemma 4.3. *Let r be large and positive. Then $A(r) < 0$.*

Proof. Provided ε is small enough, (20), (24) and the Phragmén-Lindelöf principle imply that $A(z) = O(|z|^{2\rho-1})$ for $|\arg z| \le \varepsilon$. Apply the two constants theorem to the harmonic function $u(z) = \mathrm{Re}(A(z)z^{2-2\rho})$ on the region Ω_r given by $\sqrt{r} < |z| < r^2$, $|\arg z| < \varepsilon$, and denote by c_j positive constants independent of r and ε. Let $L_0(r) = \min\{L(t) : \sqrt{r} \le t \le r^2\}$. Then $L_0(r) \ge r^{-o(1)}$ by (20). Now (24) gives

$$u(z) < -c_1 L_0(r)^2, \quad \sqrt{r} \le |z| \le r^2, \quad \arg z = \pm\varepsilon.$$

Applying the same standard estimates for harmonic measure [31, p.117] as in Lemma 3.2 shows that

$$A(r)r^{2-2\rho} = \mathrm{Re}\left(A(r)r^{2-2\rho}\right) = u(r) \le -c_1 L_0(r)^2 + O(r^{4\rho-2}r^{-c_2/\varepsilon}) < 0$$

provided ε is small enough. □

We complete the proof of Theorem 4.1 using Green's transform as in [21]. Write $E = f_1 f_2$, with f_1, f_2 linearly independent solutions of (1). Suppose with no loss of generality that f_1 has infinitely many zeros on the positive x-axis. This leads to a contradiction since

$$\frac{d}{dx}\left(f_1'(x)\overline{f_1(x)}\right) = |f_1'(x)|^2 - A(x)|f_1(x)|^2 \geq 0 \qquad (x \to +\infty).$$

This proves Theorem 4.1.

5. Proof of Theorem 1.2

Assume that (a_n) is as in the statement and that there exists a Bank-Laine function E of finite order with zero sequence (a_n). Then we may write

$$E(z) = \Pi(z)\exp(P(z) + iQ(z))$$

in which Π is a canonical product, and P, Q are real polynomials. Since E is Bank-Laine and has only real zeros, we have $\exp(iQ(z)) = \pm 1$ at every zero z of E, and so may assume that $Q \equiv 0$.

In case (i), we are assuming that $n(r) = o(r)$, and so $N(r, 1/E) = o(r)$ as $r \to \infty$. But every transcendental Bank-Laine function E has [26]

$$\liminf_{r\to\infty} \frac{T(r,E)}{r} > 0,$$

and so $\delta(0, E) = 1$, which in turn contradicts Theorem 4.1.

Now suppose that (a_n) satisfies (ii) but not (i). Then $\sum a_n^{-1}$ diverges, and we may write

$$E(z) = e^{P(z)}\Pi(z), \quad \Pi(z) = \prod_{k=1}^{\infty}(1 - z/a_k)e^{z/a_k},$$

again with P a real polynomial. We may also assume that P has degree at most 1, for otherwise a contradiction arises on applying Theorem 4.1. Now [14, p.29]

$$\log|\Pi(-r)| = \int_0^\infty \left(\frac{-r}{t} + \log\left(1 + \frac{r}{t}\right)\right) dn(t) = -r^2 \int_0^\infty \frac{n(t)}{t^2(r+t)}dt.$$

But, using [14, p.25],

$$r\int_0^\infty \frac{n(t)}{t^2(r+t)}dt \geq \frac{1}{2}\int_0^r \frac{n(t)}{t^2}dt \to \infty$$

as $r \to \infty$, since $\sum a_n^{-1}$ diverges by assumption. It follows at once that

$$\lim_{r\to\infty} \frac{T(r,E)}{r} = \lim_{r\to\infty} \frac{T(r,\Pi)}{r} = \infty$$

and again we get $\delta(0, E) = 1$, so that applying Theorem 4.1 gives a contradiction.

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