On deficiencies and fixpoints of composite meromorphic functions

D. DRASIN
Department of Mathematics, Purdue University, 47907, USA

J.K. LANGLEY
Department of Mathematics, University of Nottingham, NG7 2RD, UK

Suppose that g is a transcendental entire function and f is transcendental and meromorphic in the plane, such that the composition $F = f \circ g$ has finite order, while Q is a rational function. We prove that $F - Q$ cannot have sum of deficiencies 2. The result is related to a theorem of Goldstein and to results on the frequency of fixpoints of compositions of functions.

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1 INTRODUCTION

We begin with the following theorem of Goldstein [13], in which the notation is that of [14].

Theorem A *Suppose that f and g are transcendental entire functions, such that the composition $F = f \circ g$ has finite order. Then*

$$\sum_{a \in \mathbf{C} \cup \{\infty\}} \delta(a, F) < 2. \tag{1}$$

Theorem A is proved using results of Edrei and Fuchs [9] and Pfluger [24] concerning entire functions of finite order with near maximal deficiency sum, and indeed holds with the right hand side of (1) replaced by $2 - \delta$, with δ positive and depending on the order of F. It is clear that the hypothesis that F has finite order in Theorem A cannot simply be deleted, as the example $F = e^g$ shows at once. It is not known how large the sum of deficiencies of $f \circ g$ may be, when the composition has finite order, but examples constructed in [19] show that such functions may have infinitely many deficient values, albeit with very small deficiencies.

Our starting point is the analogue of Theorem A for meromorphic functions. The structure of meromorphic functions of finite order with $\sum \delta(a, F) = 2$ was conjectured by F. Nevanlinna, and proved by Drasin [8], with an alternative proof being given in [11] (see Section 3). We shall prove the following.

Theorem 1 *Suppose that g is a transcendental entire function and that f is transcendental and meromorphic in the plane, such that the composition $F = f \circ g$ has finite order. Then F cannot have sum of deficiencies* 2.

A large number of recent papers have discussed fixpoints of a composition $F = f \circ g$ or, more generally, zeros of $f \circ g - Q$, when Q is rational or of small growth compared to g. Bergweiler [2] proved that if f and g are transcendental entire functions, and Q is a non-constant polynomial, then $f \circ g - Q$ has infinitely many zeros. Indeed, $f \circ g$ has infinitely many repelling fixpoints [3]. Refinements and further applications of this method appear in [4, 27] and elsewhere. A different approach, based in part on a lemma of Steinmetz [25], was used in [18], in which the following was proved. If g is transcendental entire of finite lower order $\mu(g)$, while f is transcendental and meromorphic of finite order in the plane and Q is non-constant and meromorphic in the plane of order less than $\mu(g)$, then the exponent of convergence of the zeros of $f \circ g - Q$ is at least $\mu(g)$ (see also [6, 17]). With stronger hypotheses on f, g and Q, but with a stronger conclusion, the following analogue of Theorem A was proved in [20].

Theorem B *If f and g are transcendental entire functions such that the composition $F = f \circ g$ has finite order, while Q is a non-constant rational function, then $\delta(0, F - Q) < 1$.*

There do not appear to exist in the literature any examples of transcendental entire (or even meromorphic) f and g such that $\delta(0, f(g(z)) - z) > 0$. However, we prove here a result analogous to Theorem B, for meromorphic functions.

Theorem 2 *Suppose that g is a transcendental entire function and f is transcendental and meromorphic in the plane, such that the composition $G = f \circ g$ has finite order. If Q is a non-constant rational function, then $F(z) = G(z) - Q(z)$ cannot have sum of deficiencies 2.*

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2 PRELIMINARIES

We require two lemmas, the first of which is completely standard.

Lemma 1 *Let $r > 1$, let $b_1, \ldots, b_n$ be complex numbers, not necessarily distinct, each satisfying $r/4 \leq |b_j| \leq 4r$, and define $g(z) = \prod_{k=1}^{n}(1 - z/b_k)$. Then*

$$T(8r, 1/g) = T(8r, g) \leq n \log 33$$

and

$$\log |g(z)| \leq 7n \log 33 \text{ for } |z| \leq 6r.$$

Further, if $0 < \varepsilon < 1$, then

$$\log |g(z)| \geq n \log(\varepsilon/8e)$$

outside a union of discs the sum of whose radii is less than εr.

Proof The first two assertions follow immediately from the fact that $|(1 - z/b_k)| \leq 33$ on $|z| = 8r$. The Boutroux-Cartan lemma (see p.366 of [16]) gives

$$\sum_{k=1}^{n} \log |z - b_k| \geq n \log(\varepsilon r/2e)$$

outside a union of discs the sum of whose radii is less than $2e(\varepsilon r/2e) = \varepsilon r$. Thus, outside these discs we have

$$\log |g(z)| \geq n \log(\varepsilon r/2e) - \sum_{k=1}^{n} \log |b_k| \geq n \log(\varepsilon r/2e) - n \log(4r).$$

The second lemma uses a method from [6] as well as an idea of Clunie [7]. For completeness, we prove the lemma in full, although we are grateful to Walter Bergweiler for pointing out that (2) may be deduced from Satz 5.7 of [5] and the remark following it.

Lemma 2 *Suppose that f is transcendental and meromorphic in the plane and that g is a transcendental entire function, such that the composition $F = f \circ g$ has order less than $d < \infty$. Suppose further that g has lower order greater than $c > d/2$. Then*

$$\limsup_{r\to\infty} \frac{\log T(r,f)}{\log\log r} < 2. \tag{2}$$

Further, if K is a positive constant then there exist arbitrarily large R such that we have

$$f(z) = \alpha z^n(1+o(1)) \text{ for all } z \in A = \{u : K^{-1}R \le |u| \le KR\}, \tag{3}$$

in which α is an non-zero constant, and n is an integer, both possibly depending on the annulus A.

Proof Following p.358 of [6], we first note that f has order 0, by Corollary 1.1 of [10]. Thus we may write $f = f_1/f_2$, with the f_j entire of order 0, having no common zeros. Since $T(r,g) = o(T(r,F))$ (see p.54 of [14]), there is no loss of generality in assuming that $f_1(0) = f_2(0) = 1$.

We use the method of [6] to show that $f_2 \circ g$ has finite order. If δ is a sufficiently small positive constant, then Theorem 1 of [10] gives

$$n(M(r,g),f) \le n(r^{1+\delta}, f\circ g) + O(1) \le r^d \tag{4}$$

for all large r. Since f has order 0, a standard Pólya-peak argument produces arbitrarily large r such that

$$n(t,f) \le n(M(r,g),f)(t/M(r,g))^{1/2}, \quad t \ge M(r,g). \tag{5}$$

Now,

$$n(r, 1/f_2\circ g) = n(r, f\circ g). \tag{6}$$

Thus, for r satisfying (5), Theorem 1.11 of [14] gives

$$\log M(M(r,g),f_2) \le N(M(r,g),f) + n(M(r,g),f)M(r,g)^{1/2}\int_{M(r,g)}^{\infty} t^{-3/2}dt. \tag{7}$$

Since $N(t,f) \le n(t,f)\log t$ for $t > 1$, (4) and (7) give

$$\log\log M(r, f_2\circ g) \le \log\log M(M(r,g),f_2) \le O(\log r)$$

for r satisfying (5). Thus $f_2\circ g$ has finite lower order. Write $f_2 \circ g = \Pi e^h$ with Π and h both entire, the canonical product Π having order less than d, by (6). Therefore h must be a polynomial, and $f_2 \circ g$ has finite order. Indeed, by Theorem A, the order of $f_2\circ g$ is less than d. Now, for all large r, we can choose r' satisfying $r < r' = r(1+o(1))$ and lying outside a set of finite logarithmic measure, such that r' is normal for g with respect to the Wiman-Valiron theory [15], and such that the image of the disc $B(0,r')$ under g contains a circle of centre 0, radius S, with $S \ge M(r,g)$. Thus, for all large r,

$$\log M(M(r,g),f_2) \le \log M(S,f_2) \le \log M(r', f_2\circ g) \le r^d.$$

Since $M(r,g) \ge \exp(r^c)$ for all large r, (2) follows. Finally, (3) is standard (see Lemma 2 of [21], Lemma 2.1 of [20], or [1]).

3 FUNCTIONS OF FINITE ORDER WITH SUM OF DEFICIENCIES TWO

We review here some properties of functions of finite order with maximal deficiency sum, referring the reader to [8, 11, 12], but including proofs of some of these for completeness. Suppose that F is transcendental and meromorphic in the plane, of finite order ρ, with

$$\sum_{a\in\mathbf{C}} \delta(a,F) = 2. \tag{8}$$

We have assumed that $\delta(\infty,F)=0$ and we assume further that all poles of F are simple, as may be ensured by applying, if necessary, a Möbius transformation to F. It follows from (8) that

$$N(r,1/F') = o(T(r,F)). \tag{9}$$

Following Levin [22], we use the term C^0 set to denote a union of open discs $B(z_k,r_k)$ of centre z_k and radius r_k such that $z_k\to\infty$ as $k\to\infty$ and

$$\sum_{|z_k|<r} r_k = o(r) \tag{10}$$

as $r\to\infty$. By results from [8, 11, 12], 2ρ is an integer not less than 2, and there are continuous functions $L_1(r), L_2(r)$ such that

$$L_1(cr) = L_1(r)(1+o(1)), \quad L_2(cr) = L_2(r)+o(1) \tag{11}$$

as $r\to\infty$, uniformly for $1\le c\le 2$, and such that

$$T(r,F) = L_1(r)r^\rho(1+o(1)) \tag{12}$$

as $r\to\infty$. Further, there is a C^0 set C_1 such that as $r\to\infty$ we have

$$-\log|F'(re^{i\theta})| = \pi L_1(r)r^\rho(|\cos(\rho(\theta-L_2(r)))|+o(1)), \quad re^{i\theta}\notin C_1, \tag{13}$$

uniformly in θ, for $0\le\theta\le 2\pi$. We choose a large positive constant R_1 and write, for integer j with $0\le j\le 2\rho$,

$$D_j = \{z : |z|>R_1, \quad |\arg z - L_2(|z|) - \pi j/\rho| < \pi/2\rho\}. \tag{14}$$

Of course, $D_{2\rho}=D_0$. Associated with each D_j is a deficient value a_j of F and a representation for $F-a_j$ analogous to (13). For completeness, we outline how these may be obtained from (13). For each j with $0\le j<2\rho$ we can choose a path γ_j tending to infinity, parametrized by $z=\gamma_j(t)$ with $|\gamma_j(t)|$ non-decreasing, on which

$$\log|F'(z)| \le (-\pi+o(1))L_1(|z|)|z|^\rho$$

and so, for some finite a_j,

$$\log|F(z)-a_j| \le (-\pi+o(1))L_1(|z|)|z|^\rho \tag{15}$$

as $z\to\infty$ on γ_j. Let E_p be a component of the open set C_1. If E_p meets $|z|=s_1$ and $|z|=s_2$ with $s_1<s_2$ then, by (10), $s_2-s_1=o(s_2)$. Thus

$$\sum l(E_p) = o(r) \tag{16}$$

in which $l(E_p)$ denotes the sum of the radii of the discs of E_p and the sum is over those components E_p which meet $B(0,r)$. Each E_p may be enclosed in a disc of radius $2l(E_p)$, and (16) implies that the union of these discs forms a C^0 set C_2. For future reference we remark that each E_p, and hence each disc of C_2, may be assumed to contain at least one zero or pole of F', because otherwise the estimate (13) could be extended to E_p using (16) and the maximum principle.

Let $\eta(r)$ be a positive function tending to 0 as $r \to +\infty$, sufficiently slowly to satisfy certain requirements below. If

$$r > R_1, \quad |\theta - L_2(r) - \pi j/\rho| < (\pi/2\rho) - \eta(r), \tag{17}$$

and $\zeta = re^{i\theta}$ lies outside C_2, then ζ may be joined to the path γ_j by a path γ, of length $O(r)$, lying in the region

$$r(1-\eta(r)^2) < |z| < r(1+\eta(r)^2),$$
$$|\arg z - L_2(r) - \pi j/\rho| < (\pi/2\rho) - \eta(r)^2, \tag{18}$$

such that for all z on γ we have

$$\log|F'(z)| \le -\pi L_1(r)r^\rho(|\cos(\rho(\theta - L_2(r)))| + o(1)).$$

Thus we obtain

$$\log|F(re^{i\theta}) - a_j| \le -\pi L_1(r)r^\rho(|\cos(\rho(\theta - L_2(r)))| + o(1))$$

for $re^{i\theta}$ lying outside C_2 and satisfying (17). The elementary inequality

$$\log|F'(z)/(F(z)-a_j)| \le O(\log|z|)$$

which holds outside a C^0 set now gives

$$\log|F(re^{i\theta}) - a_j| = -\pi L_1(r)r^\rho(|\cos(\rho(\theta - L_2(r)))| + o(1)) \tag{19}$$

for $re^{i\theta}$ satisfying (17) and lying outside a C^0 set C_3, each disc of which contains at least one zero or pole of F' or $F - a_j$.

Lemma 3 *Let a_j be associated to the region D_j by* (19), $0 \le j < 2\rho$. *Then $a_j \ne a_{j+1}$.*

Proof Suppose that $a_j = a_{j+1} = a$. We choose positive constants ε and δ with $1/\varepsilon$ and ε/δ both large. Let r be large and let $b_1, \ldots, b_n$ be the zeros of F' in $r/4 \le |z| \le 4r$, repeated according to multiplicity. Write

$$g(z) = \prod_{k=1}^{n}(1 - z/b_k), \quad (F(z)-a)^2/F'(z) = G(z)/g(z),$$

so that G is analytic in $r/4 \le |z| \le 4r$. By Lemma 1, (9), (11) and (12) we have

$$|\log|g(z)|| = o(T(r,F)) \tag{20}$$

outside a union X_r of discs having sum of radii less than $\delta^2 r$. We can choose z_1 such that

$$(1-\delta)r \le |z_1| \le (1+\delta)r,$$
$$2\eta(r) < \arg z_1 - L_2(r) - (\pi/\rho)(j+1/2) < 4\eta(r), \tag{21}$$

with $\eta(r)$ as in (17), and such that z_1 does not lie in the C^0 set C_3 outside which (13) and (19) hold, nor the exceptional set X_r. We choose s_1 with $\varepsilon r < s_1 < 2\varepsilon r$ such that the circle T_r of centre z_1 and radius s_1 does not meet $C_3 \cup X_r$. Then (13), (19) and (20) yield

$$\log |G(z_1)| = o(T(r, F)). \tag{22}$$

On the circle T_r we have, using Lemma 1 and a standard application of the Poisson-Jensen formula,

$$\log |G(z)| \leq O(T(8r, G)) \leq c_1 T(r, F) \tag{23}$$

with c_1 an absolute constant. Further, (13) and (19) give

$$\log |G(z)| < 0 \tag{24}$$

on all of the circle T_r apart from at most two arcs each of length $o(r)$, while

$$\log |G(z)| < -\delta_1 T(r, F), \tag{25}$$

with δ_1 a fixed positive constant, on an arc of T_r of length at least $\delta_1 r$. Since

$$\log |G(z_1)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |G(z_1 + s_1 e^{i\theta})| d\theta,$$

(23), (24) and (25) contradict (22).

Lemma 4 *Let σ be a small positive constant, and let $0 \leq j < 2\rho$. Then for large r the number of zeros of $F(z) - a_j$ and $F(z) - a_{j+1}$ in the region given by*

$$r/2 \leq |z| \leq 2r, \quad |\arg z - L_2(r) - (\pi/\rho)(j + 1/2)| \leq (\pi/\rho) - \sigma$$

is $o(T(r, F))$.

Proof We choose a positive constant δ such that σ/δ is large. Define $g(z)$ exactly as in the previous lemma, and let

$$(F(z) - a_j)(F(z) - a_{j+1})/F'(z) = H(z)/g(z), \tag{26}$$

so that H is analytic in $r/4 \leq |z| \leq 4r$. We choose z_1 so that

$$(1 - \delta)r < |z_1| < (1 + \delta)r,$$

$$\delta < \arg z_1 - L_2(r) - (\pi/\rho)(j + 1/2) < 2\delta \tag{27}$$

and such that z_1 does not lie in $C_3 \cup X_r$, while the boundary ∂U of the region U given by

$$z = t z_1, \quad |\log |t|| < \log 3, \quad |\arg t| < (\pi/\rho) - 4\delta, \tag{28}$$

does not meet $C_3 \cup X_r$. On ∂U we have, using Lemma 1 and (20),

$$\log |H(z)| \leq c_2 T(r, F)$$

with c_2 an absolute constant. Further, using (13), (19) and (20), we have

$$\log |H(z)| = o(T(r, F))$$

for all z on ∂U apart from at most two arcs each of length $o(r)$, the union J of these arcs satisfying the harmonic measure estimate $\omega(z_1, J, U) = o(1)$. We map U conformally to the unit disc, with z_1 mapped to 0, and we denote the inverse mapping by $\phi(w)$. Setting $h(w) = H(\phi(w))$, we have $T(s, h) = o(T(r, F))$ for s close to 1. Since

$$\log |h(0)| = \log |H(z_1)| = o(T(r, F)),$$

by (13), (19) and (27), we obtain $N(s, 1/H) \le o(T(r, F))$ for $0 < s < 1$ and the lemma follows easily.

We close this section by remarking that similar properties hold when ∞ is one of the deficient values of F, (19) needing to be replaced by

$$\log |F(re^{i\theta})| = \pi L_1(r) r^\rho (|\cos(\rho(\theta - L_2(r)))| + o(1)) \tag{29}$$

in those regions D_j in which F is large.

4 PROOF OF THEOREM 1

Suppose then that F is transcendental and meromorphic in the plane of finite order ρ, with sum of deficiencies 2, and suppose that F can be written in the form $F = f \circ g$, with g a transcendental entire function and f transcendental meromorphic. We may assume that all deficient values of F are finite and that all poles of F are simple, applying otherwise a Möbius transformation to F and to f.

Let the functions $L_j(r)$ and regions D_j and the corresponding a_j be associated to F as in Section 3. We shall make use of the fact that in each region D_j there is a path Γ_j tending to infinity on which we have

$$\log |F(z) - a_j| < -\delta_1 T(|z|, F) \tag{30}$$

for some $\delta_1 > 0$, and this path may be chosen to contain, for arbitrarily large r, an arc T_r of $|z| = r$ of angular measure at least $\delta_2 > 0$. This may be ensured by forming Γ_j as a union of radial segments and arcs of circles. We use an idea of Goldstein [13] to observe that the image of Γ_j under g must be unbounded, for otherwise $g\Gamma_j$ tends to a finite β with $f(\beta) = a_j$, and the contribution of T_r to $m(r, 1/(F - a_j))$ is at most

$$O(1 + m(r, 1/(g - \beta))) = O(T(r, g)) = o(T(r, F)).$$

We apply the Wiman-Valiron theory to g [15]. We choose a large positive constant K and r, large and such that the central index $\nu(t) = \nu(t, g)$ satisfies $\nu(4r) \le K_1 \nu(r)$, using K_j throughout this section to denote positive constants not depending on r. We assert the following:

Lemma 5 *Let j be an integer with $0 \le j < 2\rho$. In the region*

$$V_j = \{z : 2r \le |z| \le 3r, \quad |\arg z - L_2(r) - (\pi/\rho)(j + 1/2)| < 7\pi/12\rho\}$$

we can write

$$(F(z) - a_j)/(F(z) - a_{j+1}) = G(z)H(z)$$

with $|G'(z)/G(z)| \le K_2 T(r, F)/r$ throughout V_j and

$$|\log |H(z)|| \le o(T(r, F))$$

outside at most $o(T(r,F))$ discs of total radius at most r/K^2.

Proof We first define $H(z) = \prod(1 - z/\alpha_\mu)/\prod(1 - z/\beta_\nu)$ with the products taken over all the zeros α_μ of $F - a_j$ and zeros β_ν of $F - a_{j+1}$ in the larger region U_j given by

$$U_j = \{z : r \leq |z| \leq 4r, \quad |\arg z - L_2(r) - (\pi/\rho)(j + 1/2)| < 3\pi/4\rho\}.$$

Lemmas 1 and 4 give

$$T(8r, H) + T(8r, 1/H) \leq o(T(r, F)),$$
$$T(8r, G) + T(8r, 1/G) \leq O(T(r, F))$$

and so the estimate for G'/G follows easily, since G has no zeros or poles in U_j. The estimate for H follows from Lemma 1. We denote by C_4 the union of all of the 2ρ families of exceptional discs determined by Lemma 5.

Completion of the proof of Theorem 1

We divide the interval $[2r, 3r]$ into $\nu(r)$ equal intervals of length $r/\nu(r)$. For each such interval I_p, let E_p be the measure of the set of t in I_p for which the circle $|z| = t$ meets the exceptional set C_4. There must be at least $\nu(r)/2$ of the I_p for each of which we have $E_p < r/K\nu(r)$, with K the large positive constant fixed earlier, and each of these I_p can be written in the form $[\sigma_p - r/2\nu(r), \sigma_p + r/2\nu(r)]$. The corresponding intervals $[\sigma_p - r/K\nu(r), \sigma_p + r/K\nu(r)]$ have total linear measure at least r/K. Thus, provided r is large enough, we can find some σ, normal for g with respect to the Wiman-Valiron theory [15], such that the interval $[\sigma - r/4\nu(r), \sigma + r/4\nu(r)]$ is contained in $[2r, 3r]$ and such that the following holds. The measure of the set of t for which $\sigma - r/4\nu(r) \leq t \leq \sigma + r/4\nu(r)$ and for which the circle $|z| = t$ meets the exceptional set C_4 is at most $r/K\nu(r)$. We choose z_0 with $|z_0| = \sigma$ and $|g(z_0)| = M(\sigma, g)$ and we have, for $z = z_0 e^\tau$ and $|\tau| < \nu(r)^{-2/3}$,

$$g(z) = g(z_0)(z/z_0)^N(1 + o(1)), \quad \nu(r) \leq N = \nu(\sigma, g) \leq K_1\nu(r). \tag{31}$$

Choose λ with $\sigma - r/16\nu(r) < \lambda < \sigma + r/16\nu(r)$ such that the circle $|z| = \lambda$ does not meet C_4. By (31), the image of the arc

$$J_\lambda = \{z : |z| = \lambda, \quad |\arg z - \arg z_0| \leq K_3/\nu(r)\}$$

under $\zeta = g(z)$ is a curve lying in the annulus $\{\zeta : (1 - o(1))M(\lambda, g) \leq |\zeta| \leq (1 + o(1))M(\lambda, g)\}$, and on which the maximum and minimum of $\arg\zeta$ differ by more than 2π. Further, any disc of C_4 which meets the annulus $\sigma - r/8\nu(r) < |z| < \sigma + r/8\nu(r)$ must be contained in $\sigma - r/4\nu(r) < |z| < \sigma + r/4\nu(r)$ and so any such discs have sum of radii at most $r/K\nu(r)$. This makes it possible to choose a real ϕ with $|\arg z_0 - \phi| < K_3/\nu(r)$ such that the radial segment

$$K_\phi = \{z = te^{i\phi} : \lambda - r/16\nu(r) \leq t \leq \lambda + r/16\nu(r)\}$$

does not meet C_4. The image of this radial segment under g contains a path joining the circles $|\zeta| = (1 \pm c)M(\lambda, g)$ for some small positive constant c, and on this path (31) shows that the variation of $\arg\zeta$ is small. Provided r is large enough, $A = J_\lambda \cup K_\phi$ is a connected set of diameter at most $K_4 r/\nu(r)$ whose image under g contains a closed curve γ which separates $|\zeta| < (1 - o(1))M(\lambda, g)$ from $|\zeta| > (1 + o(1))M(\lambda, g)$.

The set A must be contained in one of the overlapping regions V_j, with corresponding distinct deficient values a_j, a_{j+1} of F, and A does not meet C_4. Write $(F - a_j)/(F - a_{j+1}) = GH$ as in Lemma 5, choose any point z^* on A and assume for now that $|G(z^*)| \geq 1$. Lemma 5 now gives

$$\log|F(z) - a_j| \geq -o(T(r, F))$$

on the whole set A so that on the image $g(A)$ we have

$$\log|f(\zeta) - a_j| \geq -o(T(r, F)). \tag{32}$$

To obtain a contradiction, we recall the path Γ_j tending to infinity on which (30) holds. We have already seen that the image $g\Gamma_j$ of this path under g tends to infinity, and so must pass through the set $g(A)$ at some point $g(z)$, with $|g(z)| \geq (1 - o(1))M(\lambda, g)$ and so in particular $|z| \geq \lambda - r/16\nu(r) \geq r$, contradicting (32). The same contradiction, with $j+1$ in place of j, arises if $|G(z^*)| \leq 1$, and Theorem 1 is proved.

5 PROOF OF THEOREM 2

Suppose that Q is a non-constant rational function, that g is a transcendental entire function, and that f is transcendental and meromorphic in the plane, and suppose that $G = f \circ g$ has finite order, while

$$\sum_{a \in \mathbf{C} \cup \{\infty\}} \delta(a, F) = 2,$$

where $F(z) = G(z) - Q(z)$. By adding a constant to f, if necessary, we may assume that $Q(\infty)$ is either 0 or ∞ and that

$$Q(z) = \beta z^d(1 + o(1)), \quad \beta d \neq 0, \tag{33}$$

as z tends to infinity. We assume that the order of F is ρ and that the regions D_j, with their associated deficient values a_j, as well as the functions $L_j(r)$, are as in the discussion of Section 3. For the proof of this theorem, we cannot assume that all the deficient values are finite, although it is clear that at least one of them must be.

Lemma 6 *Suppose that the region D_j is associated with a finite deficient value a_j of F, and suppose that Γ is a path tending to infinity in D_j, on which*

$$\log|F(z) - a_j| < -\delta T(|z|, F) \tag{34}$$

for some positive constant δ. Then we have

$$\lim_{|z| \to \infty, z \in \Gamma} |g(z)| = \infty. \tag{35}$$

Proof In proving Lemma 6 there is no loss of generality in assuming that $a_j = 0$. If (35) fails to hold, there must be a finite ζ_1 such that $g(z) \to \zeta_1$ as $z \to \infty$ on Γ and such that $f(\zeta_1) = Q(\infty)$ so that

$$f(\zeta) = \nu(\zeta - \zeta_1)^m(1 + O(|\zeta - \zeta_1|)) \text{ for } 0 < |\zeta - \zeta_1| < \delta_1, \tag{36}$$

with ν a non-zero constant and m an integer. We can find arbitrarily large r such that the circle $|z| = r$ does not meet the exceptional set C_3 outside which (13) and (19) hold, and there exists $\delta_2 > 0$ such that for all such r there is some z_0 on Γ, with $|z_0| = r$ and

$$\log|F(z)| = \log|G(z) - Q(z)| < -(\delta/2)T(r, F) \tag{37}$$

on an arc Ω given by $z = z_0 e^{it}$, $\quad -\delta_2 < t < \delta_2$, with the variation of $\arg Q(z)$ on Ω at most π. We assume for now that $Q(\infty) = 0$; the proof when $Q(\infty) = \infty$ will require only trivial modifications.

By (36), there is a positive constant M such that the branch of the inverse function $\zeta = f^{-1}(w)$ which maps $G(z_0) = f(g(z_0))$ to $g(z_0)$ may be analytically continued without restriction in the annulus

$$B = \{w : 0 < |w| < M\},$$

taking values in $0 < |\zeta - \zeta_1| < \delta_1$, and satisfying

$$f^{-1}(w) - \zeta_1 = (w/\nu)^{1/m}(1 + O(|w|^{1/m})). \tag{38}$$

Further, if B_1 is a simply-connected domain contained in B, and containing $G(z_0)$, then

$$|(f^{-1})'(w)| \le c_1|w|^{-c_2}, \quad w \in B_1, \tag{39}$$

using c_j to denote non-negative constants which do not depend on r. The estimate (37) may be written in the form

$$|v^* - v| < \exp(-(\delta/2)T(r, F))$$

with $v^* = G(z)$ and $v = Q(z)$. Thus we have, for $z \in \Omega$,

$$|v| = (1 + o(1))|\beta|r^d, \quad v^* = (1 + o(1))v,$$

using (33), and (39) gives, on Ω,

$$|g(z) - f^{-1}(Q(z))| = |f^{-1}(v^*) - f^{-1}(v)| \le$$

$$\le c_3|2r|^{-dc_2} \exp(-(\delta/2)T(r, F)) \le \exp(-(\delta/4)T(r, F)). \tag{40}$$

Again using (33), $f^{-1}(Q(z))$ admits unrestricted analytic continuation in the annulus $r/4 < |z| < 4r$, starting at z_0, and these continuations satisfy there

$$\log |f^{-1}(Q(z))| \le c_4. \tag{41}$$

We can write

$$g_1(z) = g(z_0u^2), \quad f_1(u) = f^{-1}(Q(z_0u^2)),$$

these functions both analytic on the region

$$\Omega_1 = \{u : |\log |u|| < \log 2, |\arg u| < \pi\}.$$

On Ω_1 we have

$$\log |g_1(u)| \le \log M(4r, g) = o(T(r, F)). \tag{42}$$

Let $v_1(u) = \log |g_1(u) - f_1(u)|$. Then (40) gives $v_1(u) < -(\delta/4)T(r, F)$ for u lying on the arc $\Omega_2 = \{e^{it} : -\delta_2/2 \le t \le \delta_2/2\}$. Using (41), (42) and the two constants theorem (see p.42 of [23]) we easily get $g_1(u) - f_1(u) = o(1)$ for $|u| = 1$, $|\arg u| \le \pi/2$ and so $g(z) - f^{-1}(Q(z)) = o(1)$ for $|z| = r$, which, on combination with (41), clearly contradicts the fact that g is transcendental.

When $Q(\infty) = \infty$, the annulus B needs to be replaced by $\{w : M < |w| < \infty\}$, and (39) becomes $|(f^{-1})'(w)| \le c_1$. The rest is unchanged.

Lemma 7 *If the region D_j is associated with a finite deficient value a_j of F, the function g is univalent in the region $P_{8,j}$, where*

$$P_{k,j} = \{z : |z| \ge R_2/k, \quad |\arg z - L_2(|z|) - \pi j/\rho| \le k\eta_1\},$$

in which R_2 and η_1 are positive constants, with R_2 large and η_1 small.

Proof Choosing a large positive constant λ_1, we take the C^0 set C_3, a union of discs $B(q_j, p_j)$ outside which (13) and (19) hold, and we denote by C' the union of the discs $B(q_j, p'_j)$, where $p'_j = 2p_j + |q_j|^{-\lambda_1/2}$. It follows that if $|z|$ is large and z lies outside C' then the disc $B(z, |z|^{-\lambda_1})$ does not meet C_3. Because each disc of C_3 contains at least one zero or pole of F' or $F - a_k$, for some k, we see that C' is a C^0 set, at least provided λ_1 is chosen large enough. There exist positive constants η_1, η_2 such that if z_1 is large and

$$|\log |z_1/z_2|| \leq 2, \quad |\arg z_1 - \arg z_2| \leq 32\eta_1,$$

then

$$|Q(z_1) - Q(z_2)| \geq |z_1|^{-\eta_2}|z_1 - z_2|. \tag{43}$$

To obtain (43), one need only consider a branch of

$$Q_1(z) = Q(z)^{1/d} = \beta_1 z(1 + o(1)), \quad |z/z_1| > e^{-4}, \quad |\arg(z/z_1)| < 128\eta_1,$$

β_1 being a non-zero constant, and then estimate the derivative of the inverse function of Q_1.

To prove Lemma 7, suppose first that z_1 and z_2 lie in $P_{16,j}$ but outside the C^0 set C', and suppose that $G(z_1) = G(z_2)$. Then (19) gives $Q(z_1) = (1 + o(1))Q(z_2)$ and so $|\log |z_1/z_2|| = o(1)$. Thus (19) and (43) imply that

$$|z_1 - z_2| \leq \exp(-\eta_3 T(r, F)), \quad r = \min\{|z_1|, |z_2|\},$$

with η_3 a fixed positive constant. Because z_1 lies outside C', the circle T_1 of centre z_1 and radius $|z_1|^{-\lambda_1}$ does not meet C_3, and z_2 lies inside T_1. Now (19) and (43) yield

$$|Q(z) - Q(z_1)| > |(G(z) - Q(z) - a_j) - (G(z_1) - Q(z_1) - a_j)|$$

on the circle T_1. Hence $G(z) - G(z_1)$ has exactly one zero inside T_r, by Rouché's theorem, and so $z_1 = z_2$.

Now suppose that z_3 lie in the region $P_{8,j}$, and that $g(z_3) = g(z_4)$. We can choose a smooth Jordan curve L, lying in the region $P_{16,j}$, not meeting the C^0 set C', and such that z_3 and z_4 both lie inside L. Thus G is univalent on L, and therefore so is g, which implies that g is univalent inside L and $z_3 = z_4$.

Lemma 8 *With the hypotheses of Lemma 7, $g(z)$ tends to infinity as z tends to infinity in $P_{4,j}$.*

Proof We may choose paths γ_1, γ_2 tending to infinity in $P_{8,j}$, avoiding the C^0 set C_3, and intersecting each other infinitely often, in such a way that every point in $P_{4,j}$ lies either on γ_1 or γ_2, or in a bounded component of the complement of the union of the two paths. Since $g(z)$ tends to infinity as z tends to infinity on γ_1 and γ_2, by Lemma 6, and since g is univalent in $P_{8,j}$, Lemma 8 follows using Rouché's theorem.

Lemma 9 *With the hypotheses of Lemma 7, there is a positive constant N such that $|g'(z)/g(z)| \leq N/|z|$ for all z in $P_{1,j}$ with $|z|$ sufficiently large.*

Proof We may define an analytic branch of $\log g(z)$ on $P_{4,j}$. We may also choose a path γ_3, lying in $P_{4,j} \backslash P_{2,j}$ and satisfying the hypotheses of Lemma 6. Thus $g(\gamma_3) \cap g(P_{1,j})$ is empty. For z_5

in $P_{1,j}$, we now apply Bloch's theorem to $\log g(z_5 + \eta_4 |z_5| z)$, with η_4 a fixed positive constant, and this gives Lemma 9.

Completion of the proof of Theorem 2

We now argue much as in [20]. Lemma 9 gives

$$\log |g(z)| = O(\log |z|) \tag{44}$$

as z tends to infinity in $P_{1,j}$. Now, for $0 \leq j < 2\rho$, the argument of Lemmas 3 and 4 shows that ∞ cannot be the deficient value associated with both the regions D_j and D_{j+1}. This means that, for large r, we have $\log |g(z)| \leq O(\log r)$ for all z on $|z| = r$ apart from at most ρ arcs each of angular measure at most $(2\pi/\rho) - \eta_1$. Thus, a standard application of Tsuji's estimate for harmonic measure (see p.116 of [26]) shows that the lower order of g is greater than $\rho/2$, and Lemma 2 now gives (2).

We may assume that 0 is a deficient value of F. Taking a sequence s_k tending to infinity such that the circles $|z| = s_k$ do not meet the C^0 set C_3, and such that

$$2s_k \leq s_{k+1} \leq 4s_k$$

for each k, we may choose a path Γ tending to infinity in one of the $P_{1,j}$ and avoiding C_3, such that

$$\log |F(z)| < -cT(|z|, F) \tag{45}$$

as z tends to infinity on Γ, for some positive constant c, which may be chosen arbitrarily close to π. This path Γ may be formed so as to consist of radial segments joining $|z| = s_k$ to $|z| = s_{k+1}$ and, if necessary, arcs of the circles $|z| = s_k$, each of length $o(s_k)$, joining these radial segments. The function g maps γ onto a path $g\Gamma$ which tends to infinity, by Lemma 6, and Lemma 2 implies that $g\Gamma$ must pass through annuli of the form

$$A_1 = \{z : K^{-1}R \leq |\zeta| \leq KR\}, \tag{46}$$

with K a large positive constant and R arbitrarily large, on which

$$f(\zeta) = \alpha\zeta^n(1 + o(1)), \tag{47}$$

in which $\alpha \neq 0$ and n is an integer, both possibly depending on the annulus A_1. We choose z_5 on Γ such that $|g(z_5)| = R$. By Lemma 9 and the construction of Γ, we may now choose z_6 lying on Γ such that $z_6 = (1 + o(1))z_5$ and $|g(z_6)| = R(1 + o(1))$, while the circle $|z| = s = |z_6|$ does not meet C_3. Because of the choice of s, we may assume that

$$\log |F(z)| < -(c/2)T(s, F), \quad z \in \Omega' = \{u : u = z_6 e^{it}, \quad -\sigma_1 < t < \sigma_1\}, \tag{48}$$

using σ_j to denote positive constants not depending on R or s. By Lemma 9 again, $g(z)$ lies in A_1 for z lying on a subpath Γ' of Γ, which contains z_6 and joins the circles $|z| = K^{-\sigma_2}s$ and $|z| = K^{\sigma_2}s$. Because

$$f(g(z)) = (1 + o(1))Q(z) = (1 + o(1))\beta z^d, \quad d \neq 0, \tag{49}$$

on Γ, we deduce that $n \neq 0$ in (47). Consequently, a branch of the inverse function f^{-1} defined near $G(z_6)$ admits unrestricted analytic continuation in an annulus

$$A_2 = \{w : (K_1)^{-1}|\alpha|R^n \leq |w| \leq K_1|\alpha|R^n\}, \tag{50}$$

with K_1 a large constant, taking values in A_1. Further, by (49), we have $|Q(z_6)| = (1+o(1))|Q(z_5)| = (1+o(1))|G(z_5)| = (1+o(1))|\alpha|R^n$ and the image of the annulus

$$A_3 = \{z : s/4 \leq |z| \leq 4s\} \tag{51}$$

under Q lies in

$$\{w : 8^{-|d|}|\alpha|R^n \leq |w| \leq 8^{|d|}|\alpha|R^n\}$$

and so in A_2, provided K_1 was chosen large enough. In addition, with f^{-1} the same branch of the inverse as above, $f^{-1}(Q(z))$ admits unrestricted analytic continuation in A_3, starting at z_6, such that

$$|f^{-1}(Q(z))| \leq KR = (1+o(1))K|g(z_6)| \leq s^{\sigma_3} \tag{52}$$

there, using (44). Moreover, the derivative $(f^{-1})'$ may be defined on a simply-connected subdomain D^* of the annulus A_2, this subdomain containing the images of Ω' under G and Q, with Cauchy's integral formula giving

$$|(f^{-1})'(w)| \leq KRc(|\alpha|R^n)^{-1} \leq KRs^{\sigma_4}, \quad w \in D^*. \tag{53}$$

We thus have, for $z \in \Omega'$, using (48), (52) and (53),

$$\begin{aligned} &|g(z) - f^{-1}(Q(z))| = |f^{-1}(G(z)) - f^{-1}(Q(z))| \leq \\ &\leq KRs^{\sigma_3}\exp(-(c/2)T(s,F)) \leq \exp(-(c/4)T(s,F)). \end{aligned} \tag{54}$$

We define

$$g_1(u) = g(z_6u^2), \quad f_1(u) = f^{-1}(Q(z_6u^2))$$

on the simply-connected domain

$$D_1 = \{u : |\log|u|| < \log 2, \quad |\arg u| < \pi\}.$$

Since

$$\log|g_1(u)| \leq \log M(4s,g) = o(T(s,F))$$

and since (52) ensures that

$$\log|f_1(u)| \leq O(\log s)$$

on D_1, (54) and the same argument as used at the end of the proof of Lemma 6 show that $g_1(u) - f_1(u) = o(1)$, and so $\log|g(z_6u^2)| \leq O(\log s)$, for $|u| = 1, \quad |\arg u| \leq \pi/2$. This gives $\log M(s,g) = O(\log s)$ and contradicts the hypothesis that g is transcendental.

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