We may assume that 0 is a deficient value of F. Taking a sequence s_k tending to infinity such that the circles $|z| = s_k$ do not meet the C^0 set C_3, and such that

$$2s_k \leq s_{k+1} \leq 4s_k$$

for each k, we may choose a path Γ tending to infinity in one of the $P_{1,j}$ and avoiding C_3, such that

$$\log |F(z)| < -cT(|z|, F) \tag{1}$$

as z tends to infinity on Γ, for some positive constant c, which may be chosen arbitrarily close to π. This path Γ may be formed so as to consist of radial segments joining $|z| = s_k$ to $|z| = s_{k+1}$ and, if necessary, arcs of the circles $|z| = s_k$, each of length $o(s_k)$, joining these radial segments. The function g maps γ onto a path $g\Gamma$ which tends to infinity, by Lemma 6, and Lemma 2 implies that $g\Gamma$ must pass through annuli of the form

$$A_1 = \{z : K^{-1}R \leq |\zeta| \leq KR\}, \tag{2}$$

with K a large positive constant and R arbitrarily large, on which

$$f(\zeta) = \alpha\zeta^n(1 + o(1)), \quad \zeta f'(\zeta)/f(\zeta) = n + o(1), \tag{3}$$

in which $\alpha \neq 0$ and n is an integer, both possibly depending on the annulus A_1. We choose z_5 on Γ such that $|g(z_5)| = R$. By Lemma 9 and the construction of Γ, we may now choose z_6 lying on Γ such that $z_6 = (1 + o(1))z_5$ and $|g(z_6)| = R(1 + o(1))$, while the circle $|z| = s = |z_6|$ does not meet C_3. Because of the choice of s, we may assume that

$$\log |F(z)| < -(c/2)T(s, F), \quad z \in \Omega' = \{u : u = z_6 e^{it}, \quad -\sigma_1 < t < \sigma_1\}, \tag{4}$$

using σ_j to denote positive constants not depending on R or s. By Lemma 9 again, $g(z)$ lies in A_1 for z lying on a subpath Γ' of Γ, which contains z_6 and joins the circles $|z| = K^{-\sigma_2}s$ and $|z| = K^{\sigma_2}s$. Because

$$f(g(z)) = (1 + o(1))Q(z) = (1 + o(1))\beta z^d, \quad d \neq 0, \tag{5}$$

on Γ, we deduce that $n \neq 0$ in (3). Consequently, a branch ψ of the inverse function f^{-1} defined near $G(z_6)$ admits unrestricted analytic continuation in the annulus

$$A_2 = \{w : 2K^{-1}|\alpha|R^n \leq |w| \leq (1/2)K|\alpha|R^n\}, \tag{6}$$

taking values in A_1. Further, by (5), we have

$$|Q(z_6)| = (1 + o(1))|Q(z_5)| = (1 + o(1))|\alpha|R^n, \tag{7}$$

and the image of the annulus

$$A_3 = \{z : s/4 \leq |z| \leq 4s\} \tag{8}$$

under Q lies in A_2, provided K was chosen large enough. In addition, with $\psi = f^{-1}$ the same branch of the inverse as above, $f^{-1}(Q(z))$ admits unrestricted analytic continuation in A_3, starting at z_6, such that

$$|f^{-1}(Q(z))| \leq KR = (1 + o(1))K|g(z_6)| \leq s^{\sigma_3} \tag{9}$$

there, using (44) of the paper. Moreover, the derivative $\psi' = (f^{-1})'$ may be defined on a simply-connected subdomain D^* of the annulus A_2, this subdomain containing the images of Ω' under G and Q, with

$$|w\psi'(w)/\psi(w)| = |\zeta f'(\zeta)/f(\zeta)|^{-1} \leq 2,$$

and so

$$|\psi'(w)| \leq 2|\psi(w)/w| \leq 4KRK|\alpha R^n|^{-1} \leq s^{\sigma_4}, \quad w \in D^*, \tag{10}$$

by (7) and (9). We thus have, for $z \in \Omega'$, using (4) and (10),

$$|g(z) - f^{-1}(Q(z))| = |f^{-1}(G(z)) - f^{-1}(Q(z))| \leq$$

$$\leq \exp(-(c/4)T(s,F)). \tag{11}$$

We define

$$g_1(u) = g(z_6 u^2), \quad f_1(u) = f^{-1}(Q(z_6 u^2))$$

on the simply-connected domain

$$D_1 = \{u : |\log|u|| < \log 2, \quad |\arg u| < \pi\}.$$

Since

$$\log|g_1(u)| \leq \log M(4s, g) = o(T(s,F))$$

and since (9) ensures that

$$\log|f_1(u)| \leq O(\log s)$$

on D_1, (11) and the same argument as used at the end of the proof of Lemma 6 show that $g_1(u) - f_1(u) = o(1)$, and so $\log|g(z_6 u^2)| \leq O(\log s)$, for $|u| = 1, \quad |\arg u| \leq \pi/2$. This gives $\log M(s,g) = O(\log s)$ and contradicts the hypothesis that g is transcendental.