Bank-Laine functions with periodic zero-sequences

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Abstract

A Bank-Laine function is an entire function E such that $E(z) = 0$ implies that $E'(z) = \pm 1$. Such functions arise as the product of linearly independent solutions of a second order linear differential equation $w'' + A(z)w = 0$ with A entire. Suppose that

$$E(z) = R(z)e^{g(z)} \prod_{j=1}^{m} \prod_{k=1}^{q_j} (e^{\alpha_j z} - \beta_{j,k}),$$

where R is a rational function, g is a polynomial, and the α_j and $\beta_{j,k}$ are non-zero complex numbers, and that $E'(z) = \pm 1$ at all but finally many zeros z of E. Then the quotients $\alpha_j/\alpha_{j'}$ are all rational numbers and E is a Bank-Laine function and reduces to the form $E(z) = P_0(e^{\alpha z})\, e^{Q_0(z)}$ with α a non-zero complex number and P_0 and Q_0 polynomials.
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1 Introduction

A Bank-Laine function is an entire function E such that $E'(z) = \pm 1$ whenever $E(z) = 0$. These arise in connection with second order linear differential equations in the following way [1]. Let A be an entire function, and let f_1, f_2 be linearly independent solutions of

$$w'' + A(z)w = 0, \tag{1}$$

normalised so that the Wronskian satisfies $W(f_1, f_2) = 1$. Then the product $E = f_1 f_2$ is a Bank-Laine function. Conversely, every Bank-Laine function has a representation as such a product, for a suitable entire function A [3].

The use of the product $E = f_1 f_2$ to study the zeros of solutions of equation (1) was introduced in the landmark paper [1], which reinvigorated activity in this area of complex oscillation theory (see, for example, [2, 3, 4, 9, 10, 11, 13, 16]). There is a conjecture that when the exponent of convergence $\lambda(E)$ of the zeros of the product E is finite and the coefficient A is a transcendental entire function, then the order of growth $\rho(A)$ is either infinite or a positive integer. In this direction it is known that if $\rho(A)$ and $\lambda(E)$ are both finite then so is $\rho(E)$, and that if A is transcendental with $\rho(A) \leq \frac{1}{2}$ then $\lambda(E) = \infty$ [1, 17, 18] (see also [15] for an analogous result for higher order equations).

It was observed by Shen in [19] that if (a_k) is any complex sequence tending to infinity without repetition then there exists a Bank-Laine function E with zero-sequence (a_k). In the

light of the conjecture and results cited above it is then natural to ask whether there exists a Bank-Laine function of finite order with zero-sequence (a_k), in which case (a_k) is called a *Bank-Laine sequence* in the terminology of [5, 6]. Obviously every Bank-Laine sequence has finite exponent of convergence, but the converse was shown to be strongly false in [5, 6, 14]. For example, if (a_k) is an infinite sequence of non-zero real numbers and $\sum |a_k|^{-1}$ converges then (a_k) is not a Bank-Laine sequence [14].

Simple examples such as

$$\frac{e^{3z}(e^z-1)(e^z-i)}{1-i}, \quad \frac{e^{2\pi i z^2}\sin \pi z}{\pi},$$

show that the zero-sequence of a Bank-Laine function of finite order can be a union of finitely many periodic sequences. Bank-Laine functions of form $E(z) = P(e^{az})e^{bz}$, for suitable constants a and b and an appropriate choice of polynomial P, arise from certain equations (1) with a periodic entire function A as coefficient [2, 4], and for such functions the zero-sequence is again a union of periodic sequences. It then seems reasonable to ask whether it is possible for the zero-sequence of a Bank-Laine function of finite order to be a union of periodic sequences in which the periods are independent in the sense that the ratios are not all rational numbers. However, the following result shows in a strong sense that this is not the case.

Theorem 1.1 *Let $m, q_1, \dots, q_m$ be positive integers and let*

$$E(z) = R(z)e^{g(z)} \prod_{j=1}^{m} \prod_{k=1}^{q_j} \left(e^{\alpha_j z} - \beta_{j,k}\right), \tag{2}$$

where $R \not\equiv 0$ is a rational function, g is a polynomial, the α_j are pairwise distinct non-zero complex numbers, and $\beta_{j,k} \in \mathbb{C} \setminus \{0\}$ for $j = 1, \dots, m, k = 1, \dots, q_j$. Assume that

$$E'(z) = \pm 1 \text{ at all but finitely many zeros } z \text{ of } E. \tag{3}$$

Then $\alpha_j/\alpha_{j'} \in \mathbb{Q}$ for $1 \le j \le j' \le m$ and R is constant, and E is a Bank-Laine function and satisfies $E(z) = P_0\left(e^{\alpha z}\right) e^{Q_0(z)}$, with $\alpha \in \mathbb{C} \setminus \{0\}$ and P_0 and Q_0 polynomials.

Note that in (2) the $\beta_{j,k}$ must be pairwise distinct for a given j, since all but finitely many zeros of E are simple. The proof of Theorem 1.1 is based on two key facts from analytic number theory (see Lemmas 2.4 and 2.5) and a number of intermediate lemmas which may be of some independent interest.

2 Lemmas required for Theorem 1.1

The first lemma is obvious.

Lemma 2.1 *Let E be a meromorphic function satisfying condition (3). Let $a, b \in \mathbb{C}$ with $a \neq 0$, and set $F(z) = a^{-1}E(az+b)$. Then $F'(z) = \pm 1$ at all but finitely many zeros of F.*

□

Lemma 2.2 *Let $S \not\equiv 0$ be a rational function and let*

$$T(z) = \frac{S(z+1)}{S(z)}. \tag{4}$$

Suppose that $\arg T(x)$ *is constant as* $x \to +\infty$ *with* $x \in \mathbb{R}$. *Then* $\arg S(x)$ *is constant as* $x \to +\infty$ *with* $x \in \mathbb{R}$.

Proof. Since $T(\infty) = 1$ it must be the case that $\arg T(x) = 0$ as $x \to +\infty$ with $x \in \mathbb{R}$. Let

$$U(z) = \frac{\bar{S}(\bar{z})}{S(z)}, \quad V(z) = \frac{U(z+1)}{U(z)}. \tag{5}$$

Then, as $x \to +\infty$ with $x \in \mathbb{R}$,

$$V(x) = \frac{\bar{S}(x+1)}{S(x+1)}\frac{S(x)}{\bar{S}(x)} = \frac{\bar{T}(x)}{T(x)} = 1.$$

Hence $V \equiv 1$ and since a non-constant rational function cannot be periodic it follows from (5) and the fact that $U(\infty)$ has modulus 1 that there exists $\alpha \in \mathbb{R}$ such that $U \equiv e^{2i\alpha}$. Let $W(z) = e^{i\alpha}S(z)$. Then

$$\frac{\bar{W}(\bar{z})}{W(z)} = e^{-2i\alpha}U(z) \equiv 1,$$

using (5), and so $W(x)$ is real as $x \to +\infty$ with $x \in \mathbb{R}$. This proves the lemma. □

Lemma 2.3 *Let $S \not\equiv 0$ be a rational function and let Q be a polynomial. Suppose that $S(n)e^{Q(n)}$ has constant argument as $n \to +\infty$ with $n \in \mathbb{N}$. Then* $\arg S(x)$ *is constant as* $x \to +\infty$ *with* $x \in \mathbb{R}$.

Proof. The lemma will be established by induction on the degree m of the polynomial Q. Suppose first that $m = 0$. Then there is no loss of generality in assuming that $Q \equiv 0$ and that $\arg S(n) = 0$ for large $n \in \mathbb{N}$. Let U be defined by (5). Then $U(n) = 1$ for large $n \in \mathbb{N}$ and so $U \equiv 1$, from which it follows that $S(x)$ has constant argument for large positive x.

Assume now that M is a non-negative integer and that the lemma has been proved for $0 \le m \le M$, and that S and Q are as in the hypotheses, such that the polynomial Q has degree $M+1$. Set

$$Y(z) = \frac{S(z+1)e^{Q(z+1)}}{S(z)e^{Q(z)}} = T(z)e^{P(z)},$$

where T is defined as in (4) and $P(z) = Q(z+1) - Q(z)$ has degree at most M. Then $\arg Y(n)$ is constant as $n \to +\infty$ with $n \in \mathbb{N}$, and so it follows from the induction hypothesis that $\arg T(x)$ is constant as $x \to +\infty$ with $x \in \mathbb{R}$. Thus Lemma 2.2 shows that $\arg S(x)$ is constant as $x \to +\infty$ with $x \in \mathbb{R}$, and the induction is complete. □

The proof of Theorem 1.1 will require some standard facts from analytic number theory, including simultaneous approximation by rational numbers [7, p.170, Theorem 201].

Lemma 2.4 ([7]) *Let $x_1, \ldots, x_k$ and ε be real numbers, with $\varepsilon > 0$. Then there exist integers $q, p_1, \ldots, p_k$ with $q > 0$ such that $|qx_j - p_j| < \varepsilon$ for $j = 1, \ldots, k$.*

The next lemma involves the uniform distribution modulo 1 of the sequence (nx) when x is irrational [7, p.390, Theorem 445]. Here $[y]$, for $y \in \mathbb{R}$, denotes the greatest integer not exceeding y.

Lemma 2.5 ([7]) *Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Let $I \subseteq (0,1)$ be an interval of length $|I|$. Then the set*

$$F_I = \{n \in \mathbb{N} : nx - [nx] \in I\}$$

has linear density $\operatorname{dens} F_I$ *equal to $|I|$, where*

$$\operatorname{dens} F_I = \lim_{m\to\infty} \frac{\operatorname{card}(F_I \cap \{1,\dots,m\})}{m}. \tag{6}$$

The following lemma is then an immediate consequence of Lemma 2.5.

Lemma 2.6 *Let $\alpha \in \mathbb{C}$ with $\gamma = \alpha/2\pi i \notin \mathbb{Q}$. Let $\beta \in \mathbb{C} \setminus \{0\}$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that the set*

$$G_\delta = \{n \in \mathbb{N} : |e^{\alpha n} - \beta| < \delta\}$$

has upper linear density $\overline{\operatorname{dens}}\, G_\delta$ *less than ε.*

Here the upper linear density $\overline{\operatorname{dens}}\, G_\delta$ is defined as in (6) but with lim replaced by lim sup.

Proof. If γ is non-real then α has non-zero real part and so $|e^{\alpha n}|$ tends to either 0 or $+\infty$ as $n \to \infty$ with $n \in \mathbb{N}$. Thus the lemma is trivial in this case. Suppose now that γ is real but irrational, and that $n \in \mathbb{N}$. For $e^{\alpha n} - \beta$ to be small it is necessary that $\gamma n = (1/2\pi)\operatorname{Im} \alpha n$ is close to some determination of $(1/2\pi)\arg\beta$. Hence Lemma 2.6 follows from Lemma 2.5. □

3 The quotients $\alpha_j/\alpha_{j'}$ are real

Suppose that E is as in the statement of Theorem 1.1. If $1 \le j < j' \le m$ and $\alpha_j/\alpha_{j'} \in \mathbb{Q}$ then clearly there exist integers n_1, n_2 and a complex number a such that $\alpha_j = n_1 a, \alpha_{j'} = n_2 a$. The identities

$$e^{paz} - b^p = \prod_{\mu=0}^{p-1} \left(e^{az} - be^{2\pi i\mu/p}\right), \quad e^{-az} - b = -be^{-az}(e^{az} - 1/b), \tag{7}$$

are obviously valid for $p \in \mathbb{N}$ and $a, b \in \mathbb{C} \setminus \{0\}$. Thus (7) shows that there is no loss of generality in assuming throughout the rest of the paper that

$$\text{if } m \ge 2 \text{ then } \quad \frac{\alpha_j}{\alpha_{j'}} \in \mathbb{C} \setminus \mathbb{Q} \quad \text{for} \quad 1 \le j < j' \le m. \tag{8}$$

The following lemma is the first step in the proof of Theorem 1.1.

Lemma 3.1 *Let E satisfy the hypotheses of Theorem 1.1. Then the α_j satisfy*

$$\frac{\alpha_j}{\alpha_{j'}} \in \mathbb{R} \quad \textit{for} \quad 1 \le j, j' \le m. \tag{9}$$

Proof. Assume that the conclusion of the lemma is false. Then clearly $m \geq 2$ in (2). Since the α_j may be interchanged at will, there is no loss of generality in assuming that α_2/α_1 has negative imaginary part. By Lemma 2.1 it may be assumed further that

$$\alpha_1 = 2\pi i, \quad \beta_{1,1} = 1, \tag{10}$$

so that $\operatorname{Re} \alpha_2 > 0$. Write

$$g(z) = P(z) + iQ(z), \quad P, Q \in \mathbb{R}[z], \tag{11}$$

where $\mathbb{R}[z]$ denotes the set of polynomials with real coefficients. Using (2), (3), (10) and (11) there exists a non-zero constant d_1 such that

$$E(n) = 0, \quad E'(n) = \pm 1, \quad U(n) = \pm d_1 \quad \text{as } |n| \to +\infty \text{ with } n \in \mathbb{Z}, \tag{12}$$

where

$$U(n) = R(n) e^{P(n)+iQ(n)} \prod_{j=2}^{m} \prod_{k=1}^{q_j} (e^{\alpha_j n} - \beta_{j,k}) \,. \tag{13}$$

Denote positive constants by c_j. Since $\alpha_j/2\pi i \in \mathbb{C} \setminus \mathbb{Q}$ for $j \geq 2$, by (8) and (10), it follows from Lemma 2.6 that there exists a sequence (n_ν) in $\mathbb{N}$ such that $n_\nu \to +\infty$ as $\nu \to +\infty$ and

$$\left| \prod_{k=1}^{q_j} (e^{\alpha_j n_\nu} - \beta_{j,k}) \right| \geq c_1 > 0 \tag{14}$$

for each $j \geq 2$. Since α_2 has positive real part this gives

$$\left| R(n_\nu) \prod_{j=2}^{m} \prod_{k=1}^{q_j} (e^{\alpha_j n_\nu} - \beta_{j,k}) \right| \to +\infty \tag{15}$$

as $\nu \to \infty$, so that (12), (13) and (15) force

$$\lim_{n \to +\infty, n \in \mathbb{N}} P(n) = -\infty. \tag{16}$$

On the other hand simple order considerations in (13) show that P must have degree 1, since otherwise $U(n_\nu) \to 0$ by (16), contradicting (12). Hence

$$\lim_{n \to -\infty, n \in \mathbb{Z}} P(n) = +\infty. \tag{17}$$

But then Lemma 2.6 applied to the constants $-\alpha_j$ shows that there exists a sequence (n_ν) in $\mathbb{Z}$ with $n_\nu \to -\infty$ as $\nu \to \infty$ such that (14) again holds for all $j \geq 2$, which together with (13) and (17) gives $U(n_\nu) \to \infty$, contradicting (12). $\square$

4 Further lemmas

In this section a number of lemmas will be proved for functions satisfying the hypotheses of Theorem 1.1, which apply even if $m = 1$ in (2).

Let E be as in the statement of Theorem 1.1. Then (9) holds. In view of (8) and Lemmas 2.1 and 3.1 it may be assumed throughout this section that

$$\beta_{1,1} = 1, \quad \frac{\alpha_j}{2\pi i} \in \mathbb{R} \quad \text{and} \quad \frac{\alpha_j}{\alpha_{j'}} \in \mathbb{R} \setminus \mathbb{Q} \quad \text{for} \quad 1 \le j, j' \le m,\ j \ne j'. \tag{18}$$

Lemma 4.1 *Let E satisfy the hypotheses of Theorem* 1.1 *and* (18)*. Define P and Q by* (11)*. Then P is constant.*

Proof. Suppose that P is non-constant. By Lemma 2.1 with a real and $b = 0$ it may be assumed that $\alpha_1 = 2\pi i$ in addition to (18). Thus (12) again holds, where U is as in (13). Now Lemma 2.6 and (18) give a sequence of positive integers (n_ν) tending to infinity with

$$0 < c_2 < \left| \prod_{k=1}^{q_j} (e^{\alpha_j n_\nu} - \beta_{j,k}) \right| < c_3 < \infty \tag{19}$$

for each $j \ge 2$, where the c_j again denote positive constants. If P is non-constant then (13) and (19) show that $|U(n_\nu)|$ tends to one of 0 and $+\infty$ as $\nu \to +\infty$, which contradicts (12). □

Lemma 4.2 *Let E satisfy the hypotheses of Theorem* 1.1 *and* (18)*. Define P and Q by* (11)*. Then at least one of the following is true: (a) all $\beta_{j,k}$ in* (2) *have modulus* 1*; (b) the polynomial Q in* (11) *has degree at most* 1*.*

Proof. Assume that (a) and (b) are both false, so that $|\beta_{p,k}| \ne 1$ for some $p \in \{1, \dots, m\}$ and $k \in \{1, \dots, q_p\}$. Using Lemma 2.1 with a real and $b = 0$, as well as Lemma 4.1 and (18), there is no loss of generality in assuming that $\alpha_p = 2\pi i$ and $P \equiv 0$. Hence there exists a non-real complex number d such that

$$e^{\alpha_p(n+d)} = \beta_{p,k}, \quad E(n+d) = 0, \quad E'(n+d) = \pm 1 \quad \text{as } |n| \to +\infty \text{ with } n \in \mathbb{Z}. \tag{20}$$

Thus (20) gives a non-zero complex number d_2 such that

$$V(n) = \pm d_2 \quad \text{as } |n| \to +\infty \text{ with } n \in \mathbb{Z}, \tag{21}$$

where

$$V(n) = R(n+d)e^{iQ(n+d)} \prod_{1 \le j \le m,\, j \ne p} \prod_{k=1}^{q_j} \left(e^{\alpha_j(n+d)} - \beta_{j,k}\right). \tag{22}$$

By Lemma 2.6 and (18) there exists a sequence (n_ν) of positive integers tending to infinity such that

$$0 < c_4 < \prod_{1 \le j \le m,\, j \ne p} \left| \prod_{j=1}^{q_j} \left(e^{\alpha_j(n_\nu+d)} - \beta_{j,k}\right) \right| < c_5 < \infty. \tag{23}$$

Since (b) is assumed false and Q is a real polynomial and d is non-real,

$$\lim_{|n|\to+\infty, n\in\mathbb{Z}} |\mathrm{Im}\, Q(n+d)| = +\infty. \tag{24}$$

Combining (22), (23) and (24) shows that $|V(n_\nu)|$ tends to either 0 or $+\infty$ as $\nu \to +\infty$, contradicting (21). □

Lemma 4.3 *Let E satisfy the hypotheses of Theorem* 1.1 *and* (18). *Define P and Q by* (11), *and assume that Q has degree at most* 1. *Then it may be assumed without loss of generality that*

$$g(z) = P(z) + iQ(z) \equiv \alpha_{m+1} z, \tag{25}$$

where $\alpha_{m+1}/2\pi i$ is real. Moreover, $\alpha_j/\alpha_1 \in \mathbb{Q}$ for $2 \le j \le m+1$.

Proof. By Lemma 4.1 and the assumption that Q has degree at most 1 it is clearly possible to incorporate a multiplicative constant into R in (2) and thus write the polynomial g in the form (25). Further, in view of Lemma 2.1 and (18) it may be assumed again that (10) holds. Assume that $\alpha_j/\alpha_1 \notin \mathbb{Q}$ for at least one $j \ge 2$. Note that if $m \ge 2$ then $\alpha_j/\alpha_1 \notin \mathbb{Q}$ for $2 \le j \le m$ by (18). Let ε be small and positive. Then Lemma 2.4 and (18) give integers q and p_j with $q > 0$ such that

$$\frac{\alpha_j}{2\pi i} = \frac{p_j + \varepsilon_j}{q} \quad \text{where} \quad |\varepsilon_j| < \varepsilon \quad \text{for} \quad j = 2, \ldots, m+1. \tag{26}$$

By Lemma 2.1 and (10) the function $F(z) = q^{-1}E(qz)$ satisfies

$$F(n) = 0, \quad F'(n) = \pm 1 \quad \text{as } |n| \to +\infty \text{ with } n \in \mathbb{Z}. \tag{27}$$

By (2), (10), (11), (18), (25) and (26), the function F has a representation

$$F(z) = \frac{1}{q} R(qz) e^{2\pi i (p_{m+1}+\varepsilon_{m+1})z} \left(\prod_{k=1}^{q_1} \left(e^{2\pi i q z} - \beta_{1,k} \right) \right) \prod_{2\le j\le m} \prod_{k=1}^{q_j} \left(e^{2\pi i (p_j+\varepsilon_j) z} - \beta_{j,k} \right). \tag{28}$$

By (10), (27) and (28) there exists a non-zero constant d_3 such that

$$W(n) = \pm d_3 \quad \text{as } |n| \to +\infty \text{ with } n \in \mathbb{Z}, \tag{29}$$

where

$$\begin{aligned} W(z) &= \frac{1}{q} R(qz) e^{2\pi i \varepsilon_{m+1} z} \prod_{j=2}^{m} \prod_{k=1}^{q_j} \left(e^{2\pi i \varepsilon_j z} - \beta_{j,k} \right) \quad (\text{if } m \ge 2), \\ W(z) &= \frac{1}{q} R(qz) e^{2\pi i \varepsilon_2 z} \quad (\text{if } m = 1). \end{aligned} \tag{30}$$

Since $\varepsilon_j \ne 0$ for at least one $j \ge 2$, the function W is non-constant, indeed with infinitely many zeros if $m \ge 2$. But (26) and elementary estimates [8, p.7] give

$$T(r, W) \le (1 + q_2 + \ldots + q_m) 2\varepsilon r + O(\log r),$$

which contradicts (29) since ε is by assumption small. □

5 Proof of Theorem 1.1

Let E be as in the hypotheses of Theorem 1.1. Then by (8) and Lemmas 2.1 and 3.1, there is no loss of generality in assuming that (18) holds. Thus it is possible to write the polynomial g in the form (11), and to assume by Lemma 4.1 that $P \equiv 0$.

Lemma 5.1 *The integer m in (2) satisfies $m = 1$.*

Proof. By Lemmas 4.2 and 4.3 this is certainly true if any $\beta_{j,k}$ in (2) has modulus not equal to 1. Assume henceforth that $m \geq 2$, and that all $\beta_{j,k}$ in (2) have modulus 1. For s, t real,

$$e^{2isz} - e^{2it} = 2ie^{i(sz+t)} \sin(sz - t). \tag{31}$$

Since the polynomial P in (11) is by assumption identically zero it follows from (2), (18) and (31) that E may be written in the form

$$E(z) = S(z)e^{iM(z)} \prod_{j=1}^{m} \prod_{k=1}^{q_j} \sin(s_j z - t_{j,k}), \tag{32}$$

where S is a rational function, M is a real polynomial, and the s_j and $t_{j,k}$ are real constants satisfying $s_j = \alpha_j/2i$. By Lemma 2.1 it may be assumed without loss of generality that $s_1 = \pi$ and $t_{1,1} = 0$. Hence

$$E'(n) = \pm 1, \quad S(n)^2 e^{i2M(n)} \in (0, +\infty) \quad \text{as } |n| \to +\infty \text{ with } n \in \mathbb{Z}.$$

Now Lemma 2.3 shows that $\arg S(x)^2 = 2 \arg S(x)$ is constant as $x \to +\infty$ with x real. By adding a real constant to M it may thus be assumed without loss of generality that S is a real rational function. Since M is a real polynomial and all but finitely many zeros of E are real, by (32), it now follows from (3) that $e^{iM(z)} = \pm 1$ for all but finitely many zeros of E. Thus $G(z) = E(z)e^{-iM(z)}$ is such that $G'(z) = \pm 1$ at all but finitely many zeros of G. On reversing the transformation (31) and using (2), (11) and (32), it follows that G has the form

$$G(z) = T(z)e^{iN(z)} \prod_{j=1}^{m} \prod_{k=1}^{q_j} (e^{\alpha_j z} - \beta_{j,k}),$$

in which T is a rational function, N is a real polynomial of degree at most 1 and the α_j still satisfy (18). Thus Lemma 4.3 may be applied to G, which gives a contradiction, since $m \geq 2$ and (18) holds. □

In the light of Lemma 5.1 the function E may now be written in the form

$$E(z) = R(z)e^{g(z)} \prod_{k=1}^{q} (e^{\alpha z} - \beta_k),$$

and it remains to prove that the rational function R is constant and that E is a Bank-Laine function. Let $1 \leq k \leq q$. Using Lemma 2.1 again it may be assumed that $\alpha = 2\pi i$ and $\beta_k = 1$. Thus there exists a non-zero constant d_4 such that

$$E'(n) = \pm 1, \quad R(n)e^{g(n)} = \pm d_4 \quad \text{and} \quad Y(n) \in \mathbb{Z} \tag{33}$$

as $n \to +\infty$ with $n \in \mathbb{N}$, where

$$Y(z) = \frac{2\log R(z) + 2g(z) - 2\log d_4}{2\pi i}$$

is analytic of polynomial growth in a half-plane $\operatorname{Re} z > c_6 > 0$. By [12, Lemma 5] and (33), the function Y is a polynomial. Hence R is constant, since otherwise continuing analytically once around a zero or pole of R adds a non-zero integer multiple of $2\pi i$ to $\log R(z)$. Let N be a large positive integer. Then the forward differences [20, p.52]

$$\Delta Y(N) = Y(N+1) - Y(N), \quad \Delta^{p+1} Y(N) = \Delta^p Y(N+1) - \Delta^p Y(N),$$

are all integers, by (33) again. Thus $Y(\mathbb{Z}) \subseteq \mathbb{Z}$, and (33) holds for all integers n. Hence $E(z) = 0$ and $E'(z) = \pm 1$ at all zeros of $e^{\alpha z} - \beta_k$ and since this holds for each β_k it follows that E is a Bank-Laine function. This completes the proof of Theorem 1.1.

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