On differential polynomials, fixpoints and critical values of meromorphic functions

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Abstract

We prove some results concerning functions f meromorphic of finite lower order in the plane, such that $ff'' - \alpha(f')^2$ has few zeros, where α is a constant, using in part new estimates for the multipliers at fixpoints of certain functions. We go on to consider zeros of derivatives and the minimal lower growth of meromorphic functions with finitely many critical values.

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1 Introduction

The study of zeros of homogeneous polynomials in a meromorphic function and its derivatives goes back to the influential 1959 paper [18] of Hayman and beyond to the work of Milloux, Pólya and others [19]. The following theorem was proved for $k \geq 3$ in [14] and for $k = 2$ in [26] and confirmed a conjecture from [18].

Theorem A. *If f is meromorphic in the plane and f and its k'th derivative $f^{(k)}$ each have only finitely many zeros for some $k \geq 2$ then $f(z) = R(z)e^{P(z)}$, with R rational and P a polynomial.*

A stronger result was proved for $k \geq 3$ in [13]: if f is transcendental and meromorphic in the plane and, using standard notation from [19] throughout,

$$N(r, 1/ff^{(k)}) = o(T(r, f'/f)) \tag{1}$$

as $r \to \infty$, for some $k \geq 3$, then $f(z) = e^{Az+B}$, with A, B constants. It seems reasonable to conjecture that the same is true for $k = 2$, this having been proved if, in addition, f has few poles [19] or f has finite lower order [27]. A further partial result in this direction is the case $\alpha = 0$ of the following theorem [28], which also improved a result of Bergweiler [4].

Theorem B. *Suppose that f is transcendental and meromorphic in the plane, and that α is a constant.*

(i) Suppose that $\alpha \neq 1$ and $1/(\alpha - 1)$ is not a positive integer, and that

$$\limsup_{r\to\infty} \frac{N(r)}{T(r, f'/f)} = 0 \text{ and } \limsup_{r\to\infty} \frac{\log N(r)}{\log\log r} < 2, \tag{2}$$

in which $N(r) = N(r, 1/(ff'' - \alpha(f')^2))$. Then $f(z) = e^{Az+B}$, with A, B constants.

(ii) Suppose that $\alpha \neq 1$ and that $1/(\alpha - 1)$ is not an integer, or that $\alpha = 0$. If (2) holds, with $N(r)$ the counting function of the zeros of $ff'' - \alpha(f')^2$ which are not multiple zeros of f, then again $f(z) = e^{Az+B}$ with A, B constants.

Part (ii) for $\alpha = 0$ also answers completely a question of Tohge from [44, 45], by showing that the only functions f meromorphic in the plane such that f and its first two derivatives have, with finitely many exceptions, zeros at the same points, are those as above of form $f(z) = R(z)e^{P(z)}$. The differential polynomial $ff'' - \alpha(f')^2$ seems to have been considered first by Mues [37] and some results for f having few poles were proved in [43]. The proofs of Theorem B and the corresponding result in [4] are based on the use of the auxiliary function $z - hf/f'$, with $h = 1/(1-\alpha)$, so that the method breaks down when $\alpha = 1$. Indeed, as observed in [4, 28, 37], part (i) of Theorem A does not hold for $\alpha = 1$, because of examples such as $f(z) = \cos z$ or $f(z) = e^{g(z)}$ with g entire and g'' zero-free. The case where $1/(\alpha - 1)$ is a positive integer must also be excluded, due to examples of the form $f(z) = g(z)^{-n}$, with g entire such that g'' has no zeros [4]. The proof of Theorem B in [28] depends on the following result from the same paper, concerning functions with very few multiple points.

Theorem C. *Suppose that F is transcendental and meromorphic in the plane of order $\rho(F)$ satisfying $\infty \geq \rho(F) > q > 0$, and that the counting function*

$$N_1(r, F) = N(r, F) - \overline{N}(r, F) + N(r, 1/F') \tag{3}$$

of the multiple points of F satisfies

$$N_1(r, F) = o(T(r, F)), \quad \limsup_{r \to \infty} \frac{\log N_1(r, F)}{\log \log r} < 2. \tag{4}$$

Then $F(z)$ has infinitely many fixpoints z satisfying $F(z) = z, |F'(z)| > |z|^q$.

Theorem C is close to being sharp, as shown by the obvious example $F = e^P$, with P a polynomial, the proof being based in turn on a refinement of the method of [26]. The first objective of the present paper is to remove the very strong second condition of (2) and (4). Now Eremenko [11] has shown that if F has finite lower order and $N_1(r, F)$ satisfies the first condition of (4), then F has sum of Nevanlinna deficiencies

$$\sum_{a \in \mathbf{C} \cup \{\infty\}} \delta(a, F) = 2, \quad \delta(a, F) = \liminf_{r \to \infty} \frac{m(r, a, F)}{T(r, F)}, \tag{5}$$

the notation as in [19], and we use this result and the asymptotic representations proved in [7, 11] to prove the following for functions of finite lower order.

Theorem 1. *Suppose that F is meromorphic of finite lower order in the plane, with $N_1(r, F) = o(T(r, F))$ as $r \to \infty$. Then there is a positive constant c such that F has infinitely many fixpoints z with $F(z) = z$ and $|F'(z)| > cT(|z|, F)^{1/2}$ and, if $\delta(\infty, F) < 1$, the stronger estimate $|F'(z)| > c|z|T(|z|, F)^{1/2}$.*

Comparable estimates for the multipliers at fixpoints of functions in certain other classes appear in [34]. Theorem 1 gives a weaker estimate for the multiplier $F'(z)$ at fixpoints z of F than Theorem C, and the examples $\exp(z^n), \tan(z^n)$ suggest that the correct lower bound for $|F'(z)|$ might be $cT(|z|, F)$, replaced by $c|z|T(|z|, F)$ when $\delta(\infty, F) < 1$. Theorem 1 does, however, suffice for the following improvement of Theorem B when f'/f has finite lower order.

Theorem 2. *Suppose f is transcendental and meromorphic in the plane, such that f'/f has finite lower order, and suppose that α is a constant.*

(i) Suppose that $\alpha \neq 1$ and $1/(\alpha-1)$ is not a positive integer, and that

$$N(r) = o(T(r, f'/f)) \quad as \quad r \to \infty, \tag{6}$$

in which $N(r) = N(r, 1/(ff'' - \alpha(f')^2))$. Then $f(z) = e^{Az+B}$, with A, B constants.

(ii) Suppose that $\alpha \neq 1$ and that $1/(\alpha-1)$ is not an integer, or that $\alpha = 0$. If (6) holds, with $N(r)$ the counting function of the zeros of $ff'' - \alpha(f')^2$ which are not multiple zeros of f, then again $f(z) = e^{Az+B}$ with A, B constants.

In particular, if (1) holds for $k = 2$ and f'/f has finite lower order, with f transcendental and meromorphic in the plane, then $f(z) = e^{Az+B}$. The method of Theorem 2 does not apply when $\alpha = 1$, again because of the auxiliary function $H = z - hf/f', -h = 1/(\alpha - 1)$. The proof also fails when $-h$ is a positive integer, since in this case H will have multiple points at poles of f of multiplicity $-h$, although the same proof as in Theorem 2 will go through if f has no such poles.

We turn our attention now to the differential polynomial $ff'' - (f')^2$. Here we observe that $(f'/f)' = (ff'' - (f')^2)/f^2$, so that zeros of $(f'/f)'$ coincide with zeros of $ff'' - (f')^2$ which are not zeros of f. Here we have a weaker result than Theorem 2 but, using Eremenko's theorem [11] again, we prove the following.

Theorem 3. *Suppose that f is meromorphic of finite lower order in the plane, such that*

$$N(r, 1/(ff'' - (f')^2)) = o(T(r, f'/f)), \quad r \to \infty. \tag{7}$$

Then either (i)

$$f(z) = e^{Az}\left(B + e^{Cz+D}\right)^k, \tag{8}$$

or (ii)

$$f(z) = e^{Az^2+Bz+C} \quad or \quad f(z) = (Az + B)^k e^{Cz}, \tag{9}$$

in both cases with A, B, C, D constants and k an integer satisfying $k \leq 1$, or (iii) $T(r, f) = O(r)$ as r tends to infinity through a set of positive upper logarithmic density.

If, in addition, either

$$N(r, f) \leq \lambda \overline{N}(r, f) \tag{10}$$

as $r \to \infty$, for some positive constant λ, or

$$\limsup_{r\to\infty} \frac{\log^+ N(r, 1/(ff'' - (f')^2))}{\log r} < \rho(f'/f), \tag{11}$$

in which $\rho(f'/f)$ is the order of f'/f, then f satisfies either (8) or (9).

We make some remarks about Theorem 3. First, observing that $ff'' - (f')^2 = W(f, f')$, we refer the reader to [40, 41] for some related results on the zeros of the Wronskian $W(f, f', \ldots, f^{(n)})$. Second, it seems worth remarking that $T(r, f'/f)$, rather than $T(r, f)$, is the right comparison function in Theorems 2 and 3 and in (1). This is because of the existence of zero-free transcendental meromorphic functions g, of arbitrarily small growth, with poles of large multiplicity so that

$$T(r, (g'/g)') = O(T(r, g'/g)) = o(T(r, g)).$$

For such functions g, every zero of $gg'' - \alpha(g')^2$ must be a zero of $(g'/g)' + (1-\alpha)(g'/g)^2$, and thus

$$N(r, 1/(gg'' - \alpha(g')^2)) = o(T(r, g)).$$

It should be noted that in Theorem 3 it is assumed that f has finite lower order, whereas in Theorem 2 it suffices for f'/f to have finite lower order. However, examples abound of entire g of finite order such that g' has few zeros, and defining f by $f'/f = g$ will make f entire and zero-free such that $ff'' - (f')^2$ has few zeros. Indeed, the interest of Theorem 3 lies principally in the fact that, for f of infinite lower order, entire or meromorphic, even the complete absence of zeros of $ff'' - (f')^2$ does not suffice to determine f explicitly. To see this, observe that for any positive integer q we may define f by

$$(f'/f)^{(q)} = G = e^h H^{-q-1}, \tag{12}$$

in which H is an entire function, with only simple zeros, and h is an entire function, constructed using the Mittag-Leffler theorem to satisfy the following. Near each zero a of H, we have

$$G(z) = \frac{k_a(-1)^q q!}{(z-a)^{q+1}} + O(1),$$

with k_a an integer, possibly depending on a. It is then easy to see that (12) defines a meromorphic function f such that $(f'/f)^{(q)}$ has no zeros, and this may be done so that f has no zeros and the poles of f have arbitrary multiplicities $-k_a$. Alternatively, choosing $k_a = 1$ for each a makes f entire. The same construction is possible with H and h both polynomials.

As already remarked, the proof of Theorem 3 relies on observing that f'/f, with the hypotheses there, has few multiple points. Results on the zeros of f'/f itself appear in [5, 6, 12, 33]. It was proved in [6] that if the transcendental function f is meromorphic in the plane of order less than 1/2 or entire of order less than 1 then f'/f has infinitely many zeros, and both results are sharp. The same results hold with f'/f replaced by $f^{(k)}/f$, for any $k \geq 2$ [32], provided $f^{(k)}/f$ is transcendental. We prove here the following.

Theorem 4. *Let $k \geq 2$ be and let f be a meromorphic function of finite order such that $f^{(k)}/f$ is transcendental of lower order $\mu < \alpha < 1/2$. Then*

$$\delta(\infty, f) \leq \mu/\alpha \qquad \text{or} \qquad \delta(0, f^{(k)}/f) \leq 1 - \cos \pi\alpha.$$

Theorem 4 was proved in [33] for $k = 1$, but without the hypothesis that f has finite order, this assumption being needed here in order to apply certain lemmas from [22, 23]. The $\cos \pi\alpha$ term arises because the proof depends on a minimum modulus result from [17].

Our next result concerns a problem arising from Theorem A. It was conjectured in [29] that if f is meromorphic of finite order in the plane and f'' has finitely many zeros then f has finitely many poles. Examples were given in [29] showing that no such conjecture is true for functions of arbitrary growth, but the following theorem [30, 31] summarizes some progress in the direction of the conjecture.

Theorem D. *Suppose that f is transcendental and meromorphic of finite order in the plane, such that f'' has finitely many zeros, and suppose that at least one of the following conditions holds:*

(i) we have $N(r, 1/f') = o(r^{1/2})$ as $r \to \infty$;

(ii) we have $T(r, f) = O(r)$ as $r \to \infty$;

(iii) there exists a positive constant ε such that all but finitely many poles z of f have multiplicity at most $|z|^{\rho(f)-\varepsilon}$, in which $\rho(f)$ is the order of f.

Then f has finitely many poles.

The proof of Theorem D, part (ii), in [30] is complicated, relying on deep results of Hayman [21, Chapter 8] concerning subharmonic functions with maximal number of tracts. We obtain here a stronger result, with a simplified proof.

Theorem 5. *Suppose that f is transcendental and meromorphic in the plane with*

$$T(r,f) = O(r(\log r)^{\delta}) \tag{13}$$

as $r \to \infty$, in which the positive constant δ satisfies

$$e^8(200\delta)^{1/2} < 1/4, \tag{14}$$

and suppose that f'' has only finitely many zeros. Then f has only finitely many poles.

For our last result, we return to a problem considered in [27]. It was shown there that if f is meromorphic in the plane and has only finitely many critical values, these being the values taken by f at multiple points, then $T(r,f) \neq o(\log r)^2$ as $r \to \infty$. We prove here the corresponding sharp result on the lower growth of such functions.

Theorem 6. *Let f be transcendental and meromorphic in the plane, such that*

$$\liminf_{r\to\infty} \frac{T(r,f)}{(\log r)^2} = 0. \tag{15}$$

Then f has infinitely many critical values.

Theorem 6 is sharp, because of the following example from [1, 25], also referred to in [27]. There exists a meromorphic function H, constructed using the Weierstrass doubly periodic function, which satisfies a differential equation and a growth condition

$$(z^2-4)H'(z)^2 = 4(H(z)-e_1)(H(z)-e_2)(H(z)-e_3), \quad T(r,H) = O(\log r)^2 \quad \text{as} \quad r \to \infty,$$

and obviously the only critical values of H are ∞ and the distinct finite values e_1, e_2, e_3. Theorem 6 will be proved using an approximation lemma of Edrei and Fuchs [9, 15], together with a simplified version of the method of [27].

2 Lemmas needed for Theorem 1

Throughout this paper we shall use c to denote a positive constant, not necessarily the same at each occurrence. We require the following standard lemma, a proof of which, based on the Boutroux-Cartan lemma [21, p.366], may be found in [8].

Lemma A. *There are positive constants c_j such that the following is true. Let $r > 1$, let $b_1, \ldots, b_n$ be complex numbers, not necessarily distinct, each satisfying $r/16 \leq |b_j| \leq 16r$, and define G by $G(z) = \prod_{k=1}^n (1 - z/b_k)$. Then*

$$T(32r, 1/G) = T(32r, G) \leq nc_1$$

and

$$\log|G(z)| \leq nc_2 \text{ for } |z| \leq 24r.$$

Further, if $0 < \varepsilon < 1$, *then*

$$\log |G(z)| \geq n \log(\varepsilon c_3)$$

outside a union of discs the sum of whose radii is less than εr.

We next need an estimate for harmonic measure.

Lemma B [47]. *Let* $z_0 \neq 0$ *lie in the simply connected domain* D, *and let* r *be positive with* $r \neq |z_0|$. *Let* $\theta(t)$ *denote the angular measure of* $D \cap S(t)$, *where* $S(t) = \{z : |z| = t\}$, *and let* D_r *be the component of* $D \backslash S(r)$ *which contains* z_0. *Then the harmonic measure of* $S(r)$ *with respect to the domain* D_r, *evaluated at* z_0, *satisfies*

$$\omega(z_0, S(r), D_r) \leq \exp\left(\frac{-1}{\pi}\left|\int_{|z_0|}^{r} \frac{dt}{t\tan(\theta(t)/4)}\right|\right).$$

The next lemma is a standard application of harmonic measure.

Lemma 1. *Suppose that* g *is transcendental and meromorphic in the plane, and that* h, c_1, c_2, c_3 *are positive constants such that* $h > 1$ *and the following assumptions hold.*

For each r *in an unbounded set* E *there exist disjoint paths* σ_1, σ_2, *each lying in* $\{z : r \leq |z| \leq hr\}$ *and joining* $|z| = r$ *to* $|z| = hr$, *and constants* A_1, A_2, *possibly depending on* r, *with*

$$|A_1| + |A_2| \leq c_1, \quad \log |g(z) - A_j| \leq -c_2 T(h^2 r, g) \tag{16}$$

for all z *on* σ_j. *Further, there exists a domain* U, *bounded by* σ_1, σ_2, *an arc* S_1 *of* $|z| = r$ *and an arc* S_2 *of* $|z| = hr$, *such that* S_1 *and* S_2 *each have angular measure at most* $3\pi/2$ *and, for* $m = 1, 2$,

$$\log |(g(z) - A_1)(g(z) - A_2)| < o(T(h^2 r, g)) \tag{17}$$

for all z *in* $S_m \backslash T_m$, *in which each* T_m *is a sub-arc of* S_m *having angular measure* $o(1)$ *as* $r \to \infty$. *Finally, the number of poles of* g *in the set* $V = \{z : \mathrm{dist}\{z, U \cup \partial U\} \leq c_3 r\}$, *counted according to multiplicity, is* $o(T(h^2 r, g))$.

Then there exists a positive constant d *such that, for all sufficiently large* r *in* E *and for some* s *with* $h^{1/4} r < s < h^{3/4} r$, *the estimate*

$$\log |(g(z) - A_1)(g(z) - A_2)| < -dT(h^2 r, g) \tag{18}$$

holds on the intersection of U *with the circle* $|z| = s$. *In particular,* $A_1 - A_2 = o(1)$ *as* $r \to \infty$.

Proof. Let r be large, and let $b_1, \ldots, b_q$, with $q = o(T(h^2 r, g))$, be the poles of g in V, and write

$$(g(z) - A_1)(g(z) - A_2) = F(z) G(z)^{-2}, \quad G(z) = \Pi_{k=1}^{q}(1 - z/b_k),$$

so that $F(z)$ is analytic in V. Denote by $d_1, d_2, \ldots$ positive constants which do not depend on r. By Lemma A,

$$\log |G(z)| \leq o(T(h^2 r, g)) \tag{19}$$

for all z in $U \cup \partial U$, and for all such z outside a family Y_r of discs having sum of radii at most $o(r)$, we also have

$$\log |G(z)| > -o(T(h^2 r, g)). \tag{20}$$

Using (16) and (19),

$$\log |F(z)| < -d_1 T(h^2 r, g) \tag{21}$$

holds for all z in $\sigma_1 \cup \sigma_2$, and for $m = 1, 2$ we have, by (17) and (19),

$$\log |F(z)| < o(T(h^2 r, g)) \tag{22}$$

for all z in $S_m \backslash T_m$. Further, using Lemma A again we have $T(h^2 r, F) \leq (1 + o(1))T(h^2 r, g)$ and, since F is analytic in V, a standard application of the Poisson-Jensen formula gives

$$\log |F(z)| \leq d_2 T(h^2 r, g) \tag{23}$$

in $U \cup \partial U$. Choose s with $h^{1/4} r < s < h^{3/4} r$ such that the circle $|z| = s$ does not meet Y_r. For z in U with $|z| = s$ standard estimates for harmonic measure give

$$\omega(z, U, S_1 \cup S_2) \leq \omega(z, \{w : r < |w| < hr\}, S_1 \cup S_2) < d_3 < 1,$$

$$\omega(z, U, T_2) \leq \omega(z, B(0, hr), T_2) = o(1).$$

Estimating the harmonic measure of T_1 similarly, (21), (22) and (23) now give

$$\log |F(z)| \leq T(h^2 r, g)(-d_1 \omega(z, U, \sigma_1 \cup \sigma_2) + o(1)\omega(z, U, S_1 \cup S_2) + d_2 \omega(z, U, T_1 \cup T_2)) \leq -d_4 T(h^2 r, g)$$

for z in U with $|z| = s$, and (18) follows, using (20). Choosing such z with $|g(z) - A_1| = |g(z) - A_2|$ gives the second assertion of the lemma.

3 Eremenko's theorem on functions with few multiple points

Suppose that F is transcendental and meromorphic in the plane, of finite lower order, and that the counting function $N_1(r, F)$ of the multiple points of F, as defined in (3), satisfies

$$N_1(r, F) = o(T(r, F)) \text{ as } r \to \infty. \tag{24}$$

Then, by a theorem of Eremenko [11], we have (5). By results from [11] (see also [7, 42]), the order ρ of F is such that 2ρ is an integer not less than 2, and there are continuous functions $L_1(r), L_2(r)$ such that

$$L_1(cr) = L_1(r)(1 + o(1)), \quad L_2(cr) = L_2(r) + o(1) \tag{25}$$

as $r \to \infty$, uniformly for $1 \leq c \leq 2$, and such that

$$T(r, F) = L_1(r) r^\rho (1 + o(1)) \tag{26}$$

as $r \to \infty$. Further, the results of [11] give asymptotic representations for F and F' which we now describe.

Choose a large positive constant R_1 and a positive function $\eta(r)$ decreasing slowly to 0 and write, for integer j with $0 \leq j \leq 2\rho$,

$$D_j = \{z : |z| > R_1, \quad |\arg z - L_2(|z|) - \pi j/\rho| < \pi/2\rho - \eta(r)\}. \tag{27}$$

Of course, $D_{2\rho} = D_0$. Associated with each D_j are a deficient value a_j of F, finite or infinite, and representations for F' and F as follows [11], in which the standard term C^0 set [35] denotes a union of open discs $B(z_k, r_k)$ of centre z_k and radius r_k such that

$$\sum_{|z_k| < r} r_k = o(r) \tag{28}$$

as $r \to \infty$. If the deficient value a_j of F associated with D_j is finite then, for $z = re^{i\theta}$ in D_j but outside a C^0 set C_1, we have

$$\log|F'(re^{i\theta})| = -\pi L_1(r) r^\rho(|\cos(\rho(\theta - L_2(r)))| + o(1)),$$

$$\log|F(re^{i\theta}) - a_j| = -\pi L_1(r) r^\rho(|\cos(\rho(\theta - L_2(r)))| + o(1)). \tag{29}$$

On the other hand, if $a_j = \infty$ and $z = re^{i\theta}$ lies in D_j but outside C_1, then (29) is replaced by

$$\log|F'(re^{i\theta})| = \pi L_1(r) r^\rho(|\cos(\rho(\theta - L_2(r)))| + o(1)),$$

$$\log|F(re^{i\theta})| = \pi L_1(r) r^\rho(|\cos(\rho(\theta - L_2(r)))| + o(1)). \tag{30}$$

It follows easily from (28) that if E_1 denotes the set of $r > 1$ such that the circle $|z| = r$ meets the C^0 set C_1 then, for $2^n \leq r < 2^{n+1}$,

$$\int_{[1,r]\cap E_1} dt/t \leq \sum_{m=0}^{n} \int_{[2^m,2^{m+1}]\cap E_1} dt/t = o(n) = o(\log r) \tag{31}$$

as $r \to \infty$, that is, E_1 has logarithmic density 0.

The following fact is well-known and follows from the representations in [11].

Lemma C. *Let $0 \leq j \leq 2\rho - 1$. Then $a_j \neq a_{j+1}$.*

Lemma C may be proved by assuming without loss of generality that a_j and a_{j+1} are equal and finite and that all poles of F are simple, a contradiction arising on applying Lemma 1 to $(F(z) - a_j)^2/F'$, with $A_1 = A_2 = 0$, using (24) and (29). A detailed proof is given in [8].

Lemma 2. *Suppose that h is meromorphic of order 1, with*

$$\delta(0, h) = \delta(\infty, h) = 1, \tag{32}$$

and suppose that the counting function $N_1(r, h)$ defined as in (3) has order less than 1. Then h has a representation $h(z) = \Pi(z)e^{Az}$, in which Π is meromorphic of order less than 1, and A is a non-zero constant.

Proof. The hypotheses and the representations (29), (30) imply that there exist slowly varying functions $L_j(r)$ as in (25) such that

$$\log|h(re^{i\theta})| = \pi L_1(r) r(\cos(\theta - L_2(r)) + o(1)) \tag{33}$$

as $z = re^{i\theta}$ tends to infinity in $D_0 \cup D_1$ but outside a C^0 set C_1. Let σ, τ be small positive constants and let E_1 be the set of $r > 1$ such that the circle $|z| = r$ meets C_1. By (31), E_1 has logarithmic density 0. We may assume that h'/h is transcendental, for otherwise the result is obvious.

For large r not in E_1 the following is an immediate consequence of (33) and the mean value theorem. In each θ interval of length σ there is at least one θ_1 with

$$q(\theta_1) = -\pi L_1(r) r(\sin(\theta_1 - L_2(r)) + o(1)), \quad q(\theta) = \frac{\partial}{\partial\theta}(\log|h(re^{i\theta})|),$$

and, with $z_1 = re^{i\theta_1}$,

$$|z_1 h'(z_1)/h(z_1)| > r^{1-o(1)}, \tag{34}$$

using (25). It follows that for all r outside a set E_2 of upper logarithmic density at most τ we have

$$m(r, h/h') < T(r, h'/h)/2, \tag{35}$$

for otherwise a standard application of Fuchs' small arcs lemma [16] to the logarithmic derivative of h'/h would give

$$\log |h(re^{i\theta})/h'(re^{i\theta})| > c_1 T(r, h'/h)$$

for all θ in an interval of length c_2, with c_1, c_2 positive constants not depending on r, this contradicting (34) provided σ was chosen small enough.

Thus for large r not in E_2, using (35),

$$T(r, h'/h) \leq (2 + o(1))N(r, h/h') \leq (2 + o(1))N_1(r, h)$$

and so

$$N(r, 1/h) + N(r, h) \leq N_1(r, h) + T(r, h'/h) = O(N_1(r, h)).$$

Since τ may be chosen arbitrarily small it follows now that $N(r, 1/h) + N(r, h)$ has order less than 1 and the lemma is proved.

4 Auxiliary functions needed for Theorems 1 and 3

Let F and its associated deficient values a_j and regions D_j be as in the previous section. Fix j with $0 \leq j < 2\rho - 1$. Let $\{a, b\} = \{a_j, a_{j+1}\}$, with the convention that a should be finite. Define g and h by

$$h(z) = F(z) - a \quad \text{and} \quad g(z) = \frac{F(z) - a}{z - a} \quad (\text{ if } \quad b = \infty \ ),$$

$$h(z) = \frac{F(z) - a}{F(z) - b} \quad \text{and} \quad g(z) = \frac{(z - b)(F(z) - a)}{(z - a)(F(z) - b)} \quad (\text{ if } \quad b \neq \infty \ ). \tag{36}$$

Then, for large z we have $g(z) = 0$ if and only if $F(z) = a$, and $g(z) = \infty$ if and only if $F(z) = b$, while $g(z) = 1$ if and only if $F(z) = z$.

Lemma 3. *Let σ be a small positive constant, and let d be a non-zero complex number. Let $0 \leq j \leq 2\rho - 1$ and assume that g and h are defined by* (36) *and $h'/h \not\equiv d$. For large r let $n_1(r)$ be the number of zeros of $g'(z)$, with $n_2(r)$ the number of zeros of $h'(z)$, and $n_3(r)$ the number of zeros and poles of $h(z)$, and finally $n_4(r)$ the number of zeros of $d - h'(z)/h(z)$, each counted according to multiplicity, in*

$$r \leq |z| \leq 2r, \quad |\arg z - L_2(r) - (\pi/\rho)(j + 1/2)| \leq \pi/\rho - \sigma. \tag{37}$$

Then $n_1(r) + n_2(r) + n_3(r) + n_4(r) = o(T(r, F))$ as $r \to \infty$.

Proof. The assertion concerning $n_2(r)$ is an immediate consequence of (24) and (26). That concerning $n_3(r)$ is standard [7, 8, 11] but since we require a detailed proof for $n_1(r)$ we give the full argument, beginning with $n_3(r)$. Let δ, ε be positive constants, small compared to σ. Let r be large and let $b_1, \ldots, b_n$ be the poles of h/h', repeated according to multiplicity, in $r/16 \leq |z| \leq 16r$. Thus $n = o(T(r, F))$, by (24). We use M_j to denote positive constants which do not depend on r. Write

$$G(z) = \prod_{k=1}^{n} (1 - z/b_k), \quad h(z)/h'(z) = H(z)/G(z),$$

so that H is analytic in $r/16 \leq |z| \leq 16r$. By Lemma A, (24), (25) and (26) we have

$$|\log|G(z)|| = o(T(r,F)) \tag{38}$$

outside a union X_r of discs having sum of radii less than $\varepsilon^2 r$. Further, using standard estimates involving the differentiated Poisson-Jensen formula [19, p. 22],

$$|g'(z)/g(z)| + |h'(z)/h(z)| \leq |z|^{M_1} \tag{39}$$

for all z with $r/16 \leq |z| \leq 16r$ lying outside a union Y_r of discs having sum of radii less than $\varepsilon^2 r$. We can choose a path γ_r lying in

$$U^* = \{z : r \leq |z| \leq 2r, \quad |\arg z - L_2(r) - (\pi/\rho)(j+1/2)| \leq \sigma/8\}, \tag{40}$$

and joining $|z| = r$ to $|z| = 2r$, such that γ_r does not meet $X_r \cup Y_r$ nor the C^0 set C_1, and such that the length of γ_r is at most $M_2 r$ while, on γ_r,

$$\delta T(|z|,F) \leq \log|h(z)| \leq \delta(1+\delta)T(|z|,F). \tag{41}$$

Since, by (25) and (26),

$$T(2r,F) = (1+o(1))2^\rho T(r,F),$$

(41) implies that there must be a point z_1 on γ_r such that

$$r^{1+M_1} \geq |h'(z_1)/h(z_1)| \geq M_3 T(r,F)/r \geq r^{\rho-1-\sigma}, \tag{42}$$

the first inequality because of (39). We can now choose λ with

$$1 < \lambda < 1+\varepsilon \tag{43}$$

such that the boundary ∂U of the region

$$U = \{z : z = z_1 u, \quad |\log|u|| \leq \lambda \log 8, \quad |\arg u| \leq \lambda(\pi/\rho - \sigma/4)\}, \tag{44}$$

does not meet $X_r \cup Y_r \cup C_1$. On ∂U we have

$$\log^+|H(z)| = O(T(r,F)), \tag{45}$$

using Lemma A and a standard application of the Poisson-Jensen formula. Further, by (29), (30) and (36),

$$\log|h'(z)/h(z)| = o(T(r,F))$$

and so, using (38),

$$\log|H(z)| = o(T(r,F)) \tag{46}$$

on all of ∂U apart from a union T of at most 2 arcs of total length at most $M_4 r\eta(r/16)$, so that the harmonic measure $\omega(z_1,T,U)$ of T with respect to the domain U, evaluated at z_1, satisfies

$$\omega(z_1,T,U) = o(1). \tag{47}$$

Map the unit disc $|w| < 1$ to U, with 0 mapped to z_1, using a conformal mapping $\phi(w)$, which extends continuously up to $|w| = 1$. Then, for $0 < s < 1$, using (38) and (42),

$$\begin{aligned} N(s, h'(\phi(w))/h(\phi(w))) &\leq T(s, h(\phi(w))/h'(\phi(w))) + O(\log r) \\ &= T(s, H(\phi(w))/G(\phi(w))) + O(\log r) \\ &\leq T(s, H(\phi(w))) + T(s, 1/G(\phi(w))) + O(\log r) \\ &= T(s, H(\phi(w))) + T(s, G(\phi(w))) + o(T(r,F)). \end{aligned}$$

But (38), (45), (46) and (47) give

$$m(1,H(\phi(w)))+m(1,G(\phi(w)))=o(T(r,F))$$

and so, for s close to 1,

$$N(s,h'(\phi(w))/h(\phi(w)))=o(T(r,F)). \tag{48}$$

Now by (44) the mapping

$$v=(1/\lambda)\log(z/z_1)$$

maps U into

$$|Re(v)|\leq\log 8,\quad |Im(v)|\leq\pi/\rho-\sigma/4$$

and, using (43) and the fact that z_1 lies in the region U^* given by (40), v maps the region (37) into

$$|Re(v)|\leq\log 4,\quad |Im(v)|\leq\pi/\rho-\sigma/2.$$

Thus all zeros of h/h' which lie in the region (37) are mapped by ϕ^{-1} into $0<|w|\leq M_5<1$, and the assertion of Lemma 3 for $n_3(r)$ follows from (24) and (48).

The proof for $n_1(r)$ is analogous, and somewhat simpler. We estimate $n_3(r)$ as above, but with σ replaced by $\sigma/32$. Thus the number q of poles of g'/g in the region

$$U_1=\{z:r/16\leq|z|\leq 16r,\quad |\arg z-L_2(r)-(\pi/\rho)(j+1/2)|\leq\pi/\rho-\sigma/16\}$$

satisfies $q=o(T(r,F))$, and we can write

$$g'(z)/g(z)=K(z)/L(z),\quad L(z)=\Pi_{j=1}^{q}(1-z/B_j),$$

with the B_j the poles of g'/g in U_1 and $K(z)$ analytic in U_1. We have (38), with G replaced by L, and (39) implies that (46) is replaced by, on all of ∂U,

$$\log|K(z)|\leq\log|g'(z)/g(z)|+\log|L(z)|\leq O(\log r)+o(T(r,F)).$$

Since $g'(z)/g(z)=h'(z)/h(z)+O(1/|z|)$ by (36), (42) gives

$$\begin{aligned}N(s,g(\phi(w))/g'(\phi(w))) &\leq T(s,g'(\phi(w))/g(\phi(w)))+O(\log r)\\ &= T(s,K(\phi(w))/L(\phi(w)))+O(\log r)\\ &\leq m(s,K(\phi(w)))+m(s,L(\phi(w)))+o(T(r,F))\\ &= o(T(r,F))\end{aligned}$$

for s close to 1. Since zeros of g' can only arise from zeros of g'/g or multiple zeros of g, which are all multiple zeros of $F(z)-a$, the asserted estimate for $n_1(r)$ is proved.

We now turn to $n_4(r)$. First, if $\rho=1$ then (32) holds, and yields

$$n(2r,1/(d-h'/h))\leq cT(4r,h'/h)=o(T(4r,h))=o(T(r,F)).$$

We assume henceforth that $\rho>1$. Then (42) gives

$$|\log|d-h'(z_1)/h(z_1)||=O(\log r)$$

and this time we have

$$N(s,1/(d-h'(\phi(w))/h(\phi(w)))\leq T(s,h'(\phi(w))/h(\phi(w)))+O(\log r)$$

for $0 < s < 1$, the asserted estimate for $n_4(r)$ following in precisely the same way as for $n_1(r)$.

Lemma D. *For $k = j, j+1$ there are paths γ_k tending to infinity such that*

$$\arg z = L_2(|z|) + \pi k/\rho + o(1), \quad F(z) \to a_k, \quad |\log|g(z)|| \geq cT(|z|, F),$$

as z tends to infinity on γ_k.

Lemma D is an immediate consequence of (29), (30) and (36).

Lemma 4. *Let σ be a small positive constant and, for large r, let $n_5(r)$ be the number of zeros of $g(z) - 1$, counted according to multiplicity, in*

$$r/2 \leq |z| \leq 2r, \quad |\arg z - L_2(r) - (\pi/\rho)(j + 1/2)| \leq \sigma. \tag{49}$$

Then $n_5(r) > cT(r, F)$ for all sufficiently large r.

Lemma 4 follows at once from (29), (30) and (36), applying Lemma 1 to $1/(g(z) - 1)$.

We now choose large r such that the circles $|z| = r/8, r/4, 4r, 8r$ do not meet the exceptional set C_1 outside which the representations (29), (30) hold and, using Lemmas 3 and 4, simple zeros $z_1, \ldots, z_{8N}$ of $g(z) - 1$, lying in the region (49) of Lemma 4, with

$$N > cT(r, F). \tag{50}$$

Let h_m be that branch of the inverse function of $(g - 1)$ which maps 0 to z_m, and let r_m be the supremum of positive t such that h_m is analytic in $|w| < t$. Then h_m has a singularity w_m^* on $|w| = r_m$, and as t tends to 1 with $0 \leq t < 1$ the pre-image $h_m(tw_m^*)$ of tw_m^* under $g - 1$ tends either to infinity or to a multiple point of g. Let

$$r'_m = \min\{r_m, 1/4\}, \quad D'_m = h_m(B(0, r'_m)), \quad A(k) = \{z : r/k \leq |z| \leq kr\}. \tag{51}$$

Lemma 5. *At least $4N$ of the z_m are such that D'_m is contained in $A(8)$.*

Proof. Suppose that for $m = 1, \ldots, 4N$ it is the case that D'_m is not contained in $A(8)$. For these m, choose r''_m with $0 < r''_m < r'_m$ such that $D''_m = h_m(B(0, r''_m))$ meets the complement of $A(8)$. For positive t, let $\theta_m(t)$ be the angular measure of the intersection of D''_m with the circle $|z| = t$. We assert that at least $2N$ of the $D''_m, 1 \leq m \leq 4N$, are such that

$$\int_{2r}^{4r} dt/t\theta_m(t) > cN. \tag{52}$$

To see this, suppose that $D'_1, \ldots, D''_M$ are such that (52) fails. We have

$$M^2 \leq (\sum_{m=1}^{M} \theta_m(t))(\sum_{m=1}^{M} 1/\theta_m(t)), \quad M^2(\log 2)/2\pi \leq \sum_{m=1}^{M} \int_{2r}^{4r} dt/t\theta_m(t) \leq cNM,$$

and this proves our assertion. At least N of these $2N$ domains D''_m must be such that

$$\int_{r/4}^{r/2} dt/t\theta_m(t) > cN, \tag{53}$$

arguing as in (52). Using the inequality

$$1/\theta \leq (1/2)\cot(\theta/4) + 1/\pi, \quad 0 < \theta < 2\pi, \tag{54}$$

in (52) and (53), we have

$$\int_{2r}^{4r} \cot(\theta_m(t)/4)dt/t > cN, \quad \int_{r/4}^{r/2} \cot(\theta_m(t)/4)dt/t > cN, \quad m = 1, \ldots, N. \tag{55}$$

Set $\mu_m = \partial D''_m \backslash A(4)$. Now Lemma B, (50) and (55) give the harmonic measure estimate

$$\omega(z_m, D''_m, \mu_m) \leq \exp(-cT(r, F)).$$

But $\partial D''_m$ is a smooth curve on which $|g(z) - 1| = r''_m < 1/4$, and $g(z) - 1$ maps D''_m conformally onto $B(0, r''_m)$. Thus the variation of $\arg(g(z) - 1)$ on μ_m is at most $\exp(-cT(r, F))$, and there exist a simple sub-arc ν_m of $\partial D''_m$ and a constant c_m with $|c_m| = r''_m < 1/4$ such that

$$|g(z) - 1 - c_m| < \exp(-cT(r, F)) \tag{56}$$

for all z on ν_m, and such that ν_m either lies in $\{z : 4r \leq |z| \leq 8r\}$, joining $|z| = 4r$ to $|z| = 8r$, or lies in $\{z : r/8 \leq |z| \leq r/4\}$, joining $|z| = r/8$ to $|z| = r/4$. Further, by (56) and the representations (29) and (30) for F and the definition (36) of g, the arc ν_m cannot meet the paths γ_j, γ_{j+1} of Lemma D, and we must have

$$\arg z = L_2(|z|) + \pi(j + 1/2)/\rho + o(1)$$

for z on ν_m with $|z| = r/8, r/4, 4r, 8r$. By Lemma D, there exists an arc ν'_m of either γ_j or γ_{j+1} joining the same two circles on which $\log|g(z)| < -cT(r, F)$. Since g has few poles locally, by Lemma 3, we may apply Lemma 1, with the σ_j the arcs ν_m, ν'_m, and since $|1 + c_m| > 3/4$ we have a contradiction. Lemma 5 is proved.

Lemma 6. *There exist z_m with $|z_m|$ arbitrarily large and*

$$g(z_m) = 1, \quad |g'(z_m)| \geq cT(|z_m|, F)^{1/2}|z_m|^{-1}.$$

Proof. We may assume by Lemma 5 that the region D'_m is contained in $A(8)$ for $m = 1, \ldots, 4N$. Thus $r'_m = 1/4$ for at least $2N$ of these m. To establish this, we note that D'_m cannot meet the paths γ_k of Lemma D, and use Lemma 3 and (50) and the fact that, using (51), $r'_m = r_m$ implies the existence of a zero of g' on the boundary of D'_m. Now, at least N of these m are such that D'_m has area at most cr^2N^{-1} and, since g maps D'_m conformally onto $B(1, 1/4)$, we have [39, p. 4]

$$|g'(z_m)| \geq cN^{1/2}r^{-1} \geq cT(r, F)^{1/2}r^{-1},$$

using (50). This proves Lemma 6.

5 Proof of Theorem 1

To prove Theorem 1, we recall the definition (36) of g, and we apply Lemma 6, which gives

$$|F'(z_m)| \geq cT(r, F)^{1/2}$$

if $b = \infty$ and, if b is finite,

$$|F'(z_m)| \geq crT(r, F)^{1/2},$$

using the formula

$$\frac{g'}{g} = (a - b)\left(\frac{F'}{(F - a)(F - b)} - \frac{1}{(z - a)(z - b)}\right).$$

Clearly, if $\delta(\infty, F) < 1$ we may choose j so that a_j and a_{j+1}, and hence a and b, are finite.

6 Proof of Theorem 2

As in [28] we use the auxiliary function

$$H = z - hf/f', \quad h = 1/(1-\alpha), \quad H' = h(ff'' - \alpha(f')^2)/(f')^2.$$

With the hypotheses of each of parts (i) and (ii), we have $N_1(r, H) \leq N(r)$. For multiple poles of H can only occur at zeros of f' of multiplicity $m \geq 2$ which are not zeros of f, and contribute $m-1$ to each of $N_1(r, H)$ and $N(r)$, while zeros of H' cannot occur at poles of f nor, with the hypotheses of part (ii), at multiple zeros of f. However, at any fixpoint z of H we clearly have $f(z) = 0$ or $f(z) = \infty$, and $H'(z) = O(1)$. Thus H must be a rational function, with no multiple points, and so a Möbius transformation, and as in [28] we obtain $f(z) = e^{Az+B}$.

7 Proof of Theorem 3

We assume that f satisfies the hypotheses of Theorem 3, and we write $F = f'/f$ and note that all poles of F are simple. If F is rational it follows at once from (7) that F has no multiple points, and so is a Möbius transformation, in which case we easily obtain the representation (9).

We assume henceforth that F is transcendental. Since f has finite lower order, so has F and, since $N(r, 1/F') \leq N(r, 1/(ff'' - (f')^2)$, Eremenko's theorem [11] implies that F has finite order and sum of deficiencies 2. Further, $\delta(\infty, F) = 0$, because

$$m(r, F) = m(r, f'/f) = O(\log r)$$

for arbitrarily large r. We then have the estimates of Section 3 for $T(r, F)$, and may assume that the regions D_j and their associated deficient values a_j, all necessarily finite, are as in Section 3.

Lemma 7. *We have* (8)*, or*

$$N(r, 1/f) = o(T(r, F)), \quad \overline{N}(r, f) \leq o(N(r, f)) + o(T(r, F)). \tag{57}$$

Proof. Let $0 \leq j \leq 2\rho - 1$, and let h be defined by (36), with a and b necessarily finite. If h'/h is constant it follows from (36) that f satisfies (8). Suppose now that h'/h is non-constant. At a pole z of f of multiplicity m we have $F(z) = \infty$ and $h(z) = 1$ and

$$h'(z)/h(z) = (a-b)/m. \tag{58}$$

Let q be a positive integer and let $n^*(r)$ count the points at which f has zeros or poles of multiplicity at most q, each counted just once, and let $N^*(r)$ be the corresponding integrated counting function. Then we have

$$n^*(r) = o(T(r, F)), \quad N^*(r) = o(T(r, F)), \tag{59}$$

using (58), Lemma 3, and the fact that, by (25) and (26),

$$T(r, F) < c(T(r, F) - T(r/2, F)) = c\int_{r/2}^{r} A(t)dt/t < cA(r), \quad A(r) = r\frac{d}{dr}(T(r, F)).$$

Since every zero of f of multiplicity $m \geq 2$ is a zero of $ff'' - (f')^2$ of multiplicity at least m, we have

$$N(r, 1/f) \leq N^*(r) + o(T(r, F)), \quad \overline{N}(r, f) \leq N^*(r) + (1/q)N(r, f),$$

and Lemma 7 follows, using (59) and the fact that q may be chosen arbitrarily large.

Returning to the proof of Theorem 3, each estimate (29) can now be integrated again, first along a path avoiding the C^0 set C_1 outside which the representations (29) hold, such that on this path $|z|$ tends to infinity and $\arg z = L_2(|z|) + \pi j/\rho + o(1)$, and subsequently on circular arcs. This gives, using (31), with A_j a non-zero constant,

$$f(z) = A_j e^{a_j z}(1 + o(1)) \tag{60}$$

for z in D_j with $|z|$ in a set E_0 of logarithmic density 1.

Lemma 8. *We have $T(r, f) = O(r)$ as r tends to infinity through a set E_2 of positive upper logarithmic density.*

Proof. First, for r in a set E of upper logarithmic density at least 1/2 we have [20]

$$T(2r, f) = O(T(r, f)). \tag{61}$$

By (60), we have

$$|\log|f(z)|| = O(r), \quad |z| = r \in E_0, \quad \arg z \notin I_r, \tag{62}$$

in which the measure of I_r tends to 0 as $r \to \infty$. Using (61), (62) and Lemma III of [10] we obtain

$$m(r, 1/f) \leq O(r) + o(T(2r, f)) \leq O(r) + o(T(r, f)), \quad r \in E \cap E_0.$$

Now, the conclusion of Lemma 8 obviously holds if f satisfies (8). If, on the other hand, (8) does not hold then we have, using Lemma 7,

$$T(r, f) \leq O(r) + o(T(r, f)) + N(r, 1/f) \leq O(r) + o(T(r, f)) \tag{63}$$

as r tends to infinity in $E \cap E_0$ but outside a set of finite measure. This proves Lemma 8, and the first assertion of Theorem 3.

We now prove the remaining part of Theorem 3. It follows from Lemma 8 and [11] that F has order and lower order 1. Thus there are just two regions D_j, and F has two deficient values a_j, each with deficiency 1, and the function h defined by (36) satisfies (32). We will show that, subject to either of the assumptions (10), (11), we must have (8) and to this end we assume that (8) does not hold. Suppose first that (10) holds. Then (57) gives, for arbitrarily large r,

$$T(r, F) \leq \overline{N}(r, f) + \overline{N}(r, 1/f) + m(r, f'/f) \leq o(T(r, F)) + O(\log r),$$

which is plainly a contradiction. Suppose finally that (11) holds. There exists, by Lemma 8, a sequence r_n tending to infinity with $T(2r_n, f) = O(r_n)$ and, using (57),

$$m(r_n, F) = m(r_n, f'/f) = O(\log r_n), \quad \overline{N}(r_n, 1/f) + \overline{N}(r_n, f) = o(r_n).$$

Thus $T(r_n, F) = o(r_n)$ and $T(r_n, h) = o(r_n)$, whereas (11) implies that $N_1(r, h)$ has order less than 1. Using (32), this contradicts Lemma 2.

8 Proof of Theorem 4

We assume the existence of a meromorphic function f of finite order such that $f^{(k)}/f$, for some $k \geq 2$, is transcendental with lower order $\mu < \alpha < 1/2$, and we assume that

$$\delta(\infty, f) > 1 - \sigma > \mu/\alpha, \quad \delta(0, f^{(k)}/f) > 1 - \cos \pi\alpha. \tag{64}$$

We need the following lemma, in which

$$h(r) = r\frac{d}{dr}(T(r, f)) = \frac{1}{2\pi}\int_0^{2\pi} n(r, e^{i\phi}, f)d\phi \tag{65}$$

is non-decreasing, by Cartan's formula [19, p.8].

Lemma F [23]. *Let f be a transcendental meromorphic function of finite order and let L_r be any measurable subset of $[0, 2\pi)$ such that the Lebesgue measure of L_r tends to 0 as $r \to \infty$. Then there exists a subset G of $(1, +\infty)$, having logarithmic density 1, such that, as $r \to \infty$ in G,*

$$\int_{L_r} \left| Re\left(\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right)\right| d\theta = o(h(r)).$$

Lemma F is proved in [23] with

$$A(r, f) = r\frac{d}{dr}(T_0(r, f))$$

in place of $h(r)$, where $T_0(r, f)$ is the Ahlfors-Shimizu characteristic, but the version stated here admits the same proof, and follows in any case from that in [23], since by Lemma 1 of [22] we have $A(r, f) \leq 2h(r)$ off a set of finite logarithmic measure.

Lemma 9. *Choosing $K > 1$ such that*

$$\sigma < 1/K < 1 - \mu/\alpha, \tag{66}$$

there exists a subset E_1 of $(1, +\infty)$, having lower logarithmic density $1 - 1/K$, such that for r in E_1 we have

$$(1 - K\sigma)h(r) \leq I(r) = \frac{1}{2\pi}\int_0^{2\pi} \left| Re\left(\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right)\right| d\theta. \tag{67}$$

Proof. As in the proof of Theorem 1 of [33] we have, for all but isolated values of r, by simply differentiating $m(r, f) + N(r, f)$,

$$h(r) = r\frac{d}{dr}\left(T(r, f)\right) = \frac{1}{2\pi}\int_{\theta:|f(re^{i\theta})|>1} r\frac{\partial}{\partial r}\left(\log|f(re^{i\theta})|\right)d\theta + n(r, f) =$$

$$= \frac{1}{2\pi}\int_{\theta:|f(re^{i\theta})|>1} Re\left(re^{i\theta}f'(re^{i\theta})/f(re^{i\theta})\right)d\theta + n(r, f). \tag{68}$$

By (64) and (65) we have

$$\int_1^r n(t, f)dt/t \leq \int_1^r \sigma h(t)dt/t + O(1).$$

Now Lemma 3 of [2] (see also [3]) implies that there is a set E_1 of lower logarithmic density $1-1/K$ such that for all $r>1$ with $r\in E_1$ we have, using (68),

$$n(r,f)\le K\sigma h(r),\quad (1-K\sigma)h(r)\le I(r).$$

For the next lemma we need a minimum modulus result from [17], also used in [33].

Theorem E. *Suppose that g is transcendental and meromorphic in the plane, of lower order $\mu<\alpha<1$, and define*

$$G_1=\{r>1:\log L(r,g)>\gamma(\cos\pi\alpha+\delta(\infty,g)-1)T(r,g)\},$$

in which $L(r,g)=\min\{|g(z)|:|z|=r\}$ and $\gamma=\pi\alpha/\sin\pi\alpha$. Then G_1 has upper logarithmic density at least $1-\mu/\alpha$.

Lemma 10. *Let $\delta(r)$ be a positive function tending to 0 with*

$$1/\delta(r)=o(\log r) \tag{69}$$

as $r\to\infty$. There exists a subset E_2 of $(1,+\infty)$, having upper logarithmic density at least $1-\mu/\alpha$, such that for all large r in E_2 we have

$$|zf'(z)/f(z)|\le c/\delta(r) \tag{70}$$

for all z having modulus r and principal argument outside a set L_r of measure at most $c\delta(r)$.

Proof. By (64) and Theorem E, there exists a set E_2 of upper logarithmic density at least $1-\mu/\alpha$ such that for $|z|=r$ in E_2 we have

$$\log|f(z)/f^{(k)}(z)|>cT(r,f/f^{(k)}).$$

Let N be a large positive constant and let r be large, in E_2. Then we have

$$f^{(k)}(z)/f(z)=F(z),\quad |F(z)|\le r^{-N-2k},\quad |z|=r.$$

We write, for $|z|=r$,

$$f(z)=P(z)+\int_r^z\frac{(z-t)^{k-1}}{(k-1)!}f(t)F(t)dt,\quad f'(z)=P'(z)+\int_r^z\frac{(z-t)^{k-2}}{(k-2)!}f(t)F(t)dt,$$

so that

$$f(z)=P(z)+\int_r^z\varepsilon_1(z,t)f(t)dt,\quad f'(z)=P'(z)+\int_r^z\varepsilon_2(z,t)f(t)dt, \tag{71}$$

in which P is a polynomial of degree at most $k-1$ and $|\varepsilon_j(z,t)|\le r^{-N}$ for $j=1,2$. Clearly P is not the zero polynomial, since otherwise we would have $M_0=o(M_0)$ with $M_0=\max\{|f(z)|:|z|=r\}$.

Let $M=M(r,P)$. For $|z|=r$, we have, by (71),

$$|f(z)|\le v(z)=M+\int_r^z r^{-N}|f(t)|ds,$$

with s denoting arc length, and hence

$$dv/ds\le r^{-N}|f(z(s))|\le r^{-N}v(z(s)).$$

This gives $v(z) \leq 2v(r) = 2M$ for $|z| = r$ and, consequently,

$$f(z) = P(z) + o(r^{-4}M), \quad f'(z) = P'(z) + o(r^{-4}M), \quad |z| = r. \tag{72}$$

Since P has degree at most $k-1$ it follows from the Boutroux-Cartan lemma [16, 35] that

$$|P'(z)/P(z)| \leq c/r\delta(r) \tag{73}$$

for all z outside a set F_0 of discs having sum of diameters at most $r\delta(r)$. Choose ρ with $r \leq \rho \leq 2r$ such that the circle $|z| = \rho$ does not meet F_0. Then for $|z| = \rho$ we have, by (73),

$$\log |P(z)| \geq \log M(\rho, P) - c/\delta(r) \geq \log M - c/\delta(r).$$

Further, if $|w| = r$ and the principal argument of w lies outside a set of measure at most $c\delta(r)$ the ray $\arg z = \arg w$ does not meet the exceptional set F_0 and we have, integrating from the circle $|z| = \rho$ to w and using (69) and (73),

$$|P(w)| \geq Me^{-2c/\delta(r)} \geq r^{-2}M.$$

Lemma 10 follows, since using (72) and (73) we now have

$$|wf'(w)/f(w)| = |w(P'(w) + o(r^{-4}M))/(P(w) + o(r^{-4}M))| \leq c/\delta(r).$$

We may now complete the proof of Theorem 4. (66) and Lemmas 9 and 10 together give a subset E_3 of $(1, +\infty)$, having positive upper logarithmic density, such that for r in E_3 we have

$$(1 - K\sigma)h(r) \leq c/\delta(r) + J(r) = c/\delta(r) + \frac{1}{2\pi}\int_{L_r} \left| Re\left(\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right)\right| d\theta,$$

in which the measure of L_r tends to 0 as r tends to infinity in E_3. However, Lemma F gives $J(r) = o(h(r))$ as r tends to infinity in $E_3 \cap E_4$, where E_4 has logarithmic density 1. Since $1 - K\sigma > 0$, by (66), and since $\delta(r)$ may tend to 0 arbitrarily slowly, this is plainly a contradiction.

9 Proof of Theorem 5

The proof of Theorem 5 will require the following lemmas.

Lemma G [24]. *Suppose that $h(z) = \sum_{j=1}^{\infty} a_j z^j$ maps the disc $B(0, s)$ conformally onto a simply connected domain D of finite area A. Then, for real θ and $0 < r < s$, the length $L(r, \theta)$ of the image under h of the line segment $z = te^{i\theta}, 0 \leq t \leq r$, satisfies*

$$L(r, \theta)^2 \leq (A/\pi) \log 1/(1 - r^2 s^{-2}).$$

Lemma H [30]. *Let M, N be integers with $0 \leq M$ and $N > 2(M + 2)$. Let $r > 1$ and let p be a positive integer with $p \leq r^2$. Let $0 < s \leq r^{-N}$, and let D_j be the closed disc $\{z : |z - \omega_j| \leq s\}$, $\omega_j = \exp(\pi i j/p)$, for $j = 1, \ldots, 2p$. Let E be a subset of the unit circle $T = \{z : |z| = 1\}$, a union of finitely many closed arcs, having arc length $m(E) \geq r^{-2}$ and not meeting any of the closed discs D_j. Finally, let $U = \Delta \backslash (\bigcup_{j=1}^{2p} D_j)$, and let z_0 be any point in Δ with $|z_0| \leq 1 - r^{-M}$. Then,* *provided r is large enough, the harmonic measure of E with respect to the domain U, evaluated at z_0, satisfies*

$$\omega(z_0, E, U) \geq (1/8\pi) m(E) r^{-M}.$$

Lemma J [33]. *Suppose that F is transcendental and meromorphic in the plane with $T(r, F) = O(r^{3/2})$ as $r \to \infty$. Then there exist arbitrarily large positive R such that the length $L(r, R)$ of the level curves $|F(z)| = R$ lying in $|z| \leq r$ satisfies $L(r, R) = O(r^{9/4})$ as $r \to \infty$.*

Suppose now that f satisfies the hypotheses of Theorem 5, but has infinitely many poles. We first note that, as proved in [30], the function $g = f'$ has at least two distinct finite asymptotic values a_1 and a_2, with paths Γ_j tending to infinity on which $g(z)$ tends to a_j. A result of Bergweiler and Eremenko [5] implies that, since g has finite order and g' has only finitely many zeros, the inverse function g^{-1} cannot have an indirect transcendental singularity [5, 38]. Thus the asymptotic paths Γ_j correspond to direct transcendental singularities of g^{-1} over a_j and, because f has order at most 1, there cannot be any other direct transcendental singularities. We may assume that $a_j = (-1)^j$. By the definition of a direct singularity [38], we can take a small positive b_1 such that each path Γ_j lies in a simply connected component U_j of the set $\{z : |g(z) - a_j| < b_1\}$ on which $g(z) \neq a_j$.

Further, by the discussion in [30], all but finitely many components U^* of the set $\{z : |g(z)| < 1\}$ are of the following form. U^* is simply connected and conformally equivalent under g to the unit disc, with boundary consisting of either one simple curve going to infinity in both directions and joining one of U_1, U_2 to itself (type I in the notation of [30]), or (type II) two simple curves each going to infinity in both directions and joining U_1 to U_2. Further, all but finitely many components V^* of the set $\{z : |g(z)| > 1\}$ are such that V^* is simply connected and contains precisely one pole of g of multiplicity q, with V^* conformally equivalent under $g(z)^{-1/q}$ to the unit disc: moreover, V^* borders on two type II components of the set $\{z : |g(z)| < 1\}$.

Returning to the regions U_j, we may define corresponding to each a function H_j, subharmonic in the plane, by $H_j(z) = \log |b_1/(g(z) - a_j)|$ for z in U_j and $H_j(z) = 0$ elsewhere. It follows that U_j contains a path λ_j tending to infinity, on which $H_j(z)/\log |z| \to \infty$. We now need to choose regions V_j contained in the U_j such that the integral of $|g(z) - a_j|$ over the boundary of V_j converges. To do this, we first assume that $|g(z)| \leq 1/2$ for $|z| \leq 2$, replacing f if necessary by $f(A_1 z + B_1)/A_1$ for suitable constants A_1, B_1. We choose S_j with $0 < S_j < b_1$ such that the total length of the level curves $|z^4(g(z) - a_j)| = S_j$ lying in $|z| \leq r$ is $O(r^{9/4})$ as $r \to \infty$, applying Lemma J to $1/z^4(g(z) - a_j)$. We have $|z^4(g(z) - a_j)| > S_j$ on $|z| = 2$ and, if S^* denotes the union of the level curves $|z^4(g(z) - a_j)| = S_j, j = 1, 2$, lying in $|z| \geq 2$ then, for some large n_0,

$$\int_{S^*} |g(z) - a_j||dz| < O(1) + \sum_{n=n_0}^{\infty} S_j 2^{-4n} 2^{3(n+1)} < \infty. \tag{74}$$

Set $h_j(z) = S_j z^{-4}(g(z) - a_j)^{-1}$. This function is analytic on U_j. Further, we have $|h_j(z)| < 1$ on the boundary of U_j but U_j contains at least one component of the set $\{z : \log |h_j(z)| > 0\}$, containing at least part of the path λ_j. If we define $U(z)$ to be 0 outside the union of U_1 and U_2 and $\log^+ |h_j(z)|$ in U_j, then U is subharmonic in the plane and

$$m(r, U) \leq m(r, 1/(g - a_1)) + m(r, 1/(g - a_2)) + O(\log r) \leq r^{1+o(1)}$$

as $r \to \infty$. This growth estimate, together with the fact that each U_j contains at least one component of the set $\{z : U(z) > 0\}$, implies that each U_j contains precisely one component V_j of the set $\{z : \log |h_j(z)| > 0\}$. Moreover, since $h_j(z) \neq 0, \infty$ on U_j, the component V_j is unbounded and simply connected.

We estimate f on each V_j. Let z_1 and z_2 be arbitrary points in V_1. We join z_1 to z_2 by a path consisting of part of the ray $\arg z = \arg z_1$ followed by part of the circle $|z| = |z_2|$, and then replace any parts of this path which leave V_1 by arcs of the boundary of V_1. Using (74), we thus have $f(z_1) + z_1 = f(z_2) + z_2 + O(1)$ and so, fixing z_2, we deduce that

$$f(z) = a_j z + O(1), \quad z \in V_j. \tag{75}$$

By arguments similar to those above, if the positive constant b_2 is chosen small enough, the region V_j contains precisely one component W_j of the set $\{z : |(g(z) - (-1)^j)z^{17}| < b_2\}$, and W_j is unbounded. We denote the angular measure of the intersection of W_j with the circle $|z| = r$, for r large, by $\theta_j(r)$. We define $u_1(z)$, subharmonic in the plane, by

$$u_1(z) = \log |b_2/(g(z) - (-1)^j)z^{17}|, \quad z \in W_j, \tag{76}$$

and $u_1(z) = 0$ elsewhere, and we have, as $r \to \infty$,

$$T(r, u_1) \leq m(r, 1/(f' - 1)) + m(r, 1/(f' + 1)) + O(\log r) = O(r(\log r)^\delta). \tag{77}$$

Now (77) and the standard estimate for harmonic measure due to Tsuji [46, p.116] give

$$\sum_{j=1}^{2} \int_{r_1}^{r} \pi dt/t\theta_j(t) \leq 2 \log B(2r, u_1) + O(1) \leq 2 \log 4r + 2\delta \log\log 4r + O(1)$$

for some $r_1 > 0$ and for all large r. But

$$\pi/\theta_1(t) + \pi/\theta_2(t) \geq 4\pi/(\theta_1(t) + \theta_2(t)) = 4\pi/(2\pi - \theta(t)) \geq 2 + \theta(t)/\pi,$$

in which $\theta(t) = 2\pi - \theta_1(t) - \theta_2(t)$, and so

$$\int_{r_1}^{r} \theta(t)dt/t < 3\delta\pi \log\log r, \quad r \to \infty. \tag{78}$$

By (78) there exist arbitrarily large positive r such that

$$\int_{e^{-4}r}^{e^4 r} \theta(t)dt/t < 4\delta\pi(\log\log(e^4 r) - \log\log(e^{-4}r)) < 40\delta\pi/\log r. \tag{79}$$

Using the Cauchy-Schwarz inequality, writing $1/t = (\theta(t)/t)^{1/2}(1/t\theta(t))^{1/2}$, we thus have

$$\log r < 40\delta\pi \left(\int_s^{es} dt/t\theta(t) \right)$$

and hence, using (54),

$$\log r < 30\delta\pi \left(\int_s^{es} \frac{dt}{t \tan(\theta(t)/4)} \right) \tag{80}$$

for $e^{-4}r \leq s \leq e^3 r$. Define $A(q) = \{z : e^{-q}r \leq |z| \leq e^q r\}$.

Lemma 11. *Provided r is large enough and satisfies (79), the annulus $A(2)$ contains at least one pole z_3 of f'.*

Proof. Suppose on the contrary that $A(2)$ contains no pole of f'. Choose s_1 and s_2 with

$$e^{-2}r \leq s_1 \leq e^{-1}r, \quad er \leq s_2 \leq e^2 r,$$

such that $\log |f'(z)^2 - 1| \leq r^2$ on $|z| = s_j$, for $j = 1, 2$, this possible by a standard application of the Poisson-Jensen formula. There exists a domain D_0 in the complement of $W_1 \cup W_2$, bounded by an arc of ∂W_1 on which $f'(z) = -1 + o(1)$ and an arc of ∂W_2 on which $f'(z) = 1 + o(1)$, as well as by arcs T_j of the circles $|z| = s_j$, for $j = 1, 2$, such that if z is in D_0 with $|z| = r$ then (80) gives

$$\omega(z, T_1, D_0) \leq \exp\left(-\frac{1}{\pi}\int_{r/e}^{r} \frac{dt}{t\tan(\theta(t)/4)}\right) \leq \exp(-(1/30\delta\pi^2)\log r),$$

using Lemma B. Estimating the harmonic measure of T_2 similarly, we get

$$\log |f'(z)^2 - 1| < \log o(1) + 2r^2 \exp(-(1/30\delta\pi^2)\log r) < \log o(1),$$

using (14), and so $f'(z)^2 - 1 = o(1)$ on the intersection μ of $|z| = r$ with D_0. Choosing z on μ with $|f'(z) - 1| = |f'(z) + 1|$ we obtain a contradiction in the standard way. This proves Lemma 11.

Returning to the proof of Theorem 1, we choose large r satisfying (79) and denote the multiplicity of the pole of f' at z_3 by p. We use c_j to denote positive constants not depending on δ or r. Now z_3 lies in a simply connected region D which is a component of the set

$$\{z : |f'(z)| > 1, \quad |f'(z) - 1| > |z_3|^{-16}, \quad |f'(z) + 1| > |z_3|^{-16}\}$$

and D does not intersect W_1 or W_2 within $A(6)$. Provided r is large enough, there is a component E of the set

$$\{z : |f'(z)| < 1, \quad |f'(z) - 1| > |z_3|^{-16}, \quad |f'(z) + 1| > |z_3|^{-16}\},$$

contained in a type II component of $\{z : |f'(z)| < 1\}$, such that the boundaries of D and E share an arc J on which $|f'(z)| = 1, |\arg f'(z) - \pi/2| \leq \pi/16$, this arc J forming part of an unbounded level curve L_1 on which $|f'(z)| = 1$, joining U_1 to U_2. The function $v(z) = (f'(z))^{-1/p}$ is analytic and univalent on D, mapping D onto a domain D^*, such that D^* contains the domain D^{**} consisting of the unit disc with circular indentations at the $2p$ roots $\exp(\pi ij/p), 1 \leq j \leq 2p$, these indentations having radius $r_2 \leq |z_3|^{-15}$. The arc J is mapped by v into an arc J^* of the unit circle of length at least $\pi/8p \geq c_1|z_3|^{-3/2}$, so that Lemma H gives

$$\omega(z_3, J, D) = \omega(0, J^*, D^*) \geq \omega(0, J^*, D^{**}) \geq c_2/p \geq c_3|z_3|^{-3/2}. \tag{81}$$

Let J_1 be the part of J which lies in $|z| < e^{-3}r$, if any, and denote by D_1 the component of $D \cap \{z : |z| > e^{-3}r\}$ which contains z_3. Then

$$\omega(z_3, J_1, D) \leq \omega(z_3, D \cap \{z : |z| = e^{-3}r\}, D_1) \leq$$

$$\leq \exp\left(\frac{-1}{\pi}\int_{e^{-3}r}^{|z_3|} \frac{dt}{t\tan(\theta(t)/4)}\right) \leq \exp(-(1/30\pi^2\delta)\log r),$$

using Lemma B and (80). Estimating the harmonic measure at z_3 of any part of J lying in $|z| > e^3 r$ similarly, and using (14), we deduce the following. There is a subset J' of J, a union of finitely many closed arcs of the level curve L_1, such that J' lies in $A(3)$ and

$$\omega(z_3, J', D) \geq c_4/p.$$

This means that the variation of $\arg f'(z)$ on J' must be at least c_5.

We now consider the harmonic measure of J' in E, which is mapped univalently by f' into the unit disc with circular indentations of radius $|z_3|^{-16}$ at 1 and -1. Let

$$L = \{z \in E : |f'(z)| \leq 1 - |z_3|^{-5}\}.$$

Then L does not meet W_1 or W_2 in $A(6)$. Lemma H applies again, to give

$$\omega(z, J', E) \geq c_6|z_3|^{-5} \geq c_7 r^{-5}, \quad z \in L. \tag{82}$$

But since J' lies in $A(3)$, L must lie in $A(4)$. For otherwise, if z lies in L but outside $A(4)$, a further application of Lemma B and (80) gives

$$\omega(z, J', E) \leq 2\exp(-(1/30\pi^2\delta)\log r),$$

which, by (14), contradicts (82). Also, by (79), the area of L is at most

$$e^8 r^2 40\delta\pi/\log r.$$

Let γ be the pre-image in E under f' of the interval $[-1 + |z_3|^{-9/2}, 1 - |z_3|^{-9/2}]$. Then γ lies in L and Lemma G implies that the length S of γ satisfies

$$S^2 \leq (e^8 r^2 40\delta/\log r) 5\log r = 200e^8\delta r^2. \tag{83}$$

Now γ joins a point ζ_1 in U_1 to a point ζ_2 in U_2, and since γ lies in L and so in $A(4)$ we have

$$|f'(\zeta_j) - (-1)^j| = o(|z_3|^{-4}) = o(|\zeta_j|^{-4})$$

so that ζ_j lies in V_j. We thus have, on the one hand, using (14) and (83),

$$|\zeta_2 - \zeta_1| \leq e^4(200\delta)^{1/2} r \leq e^8(200\delta)^{1/2}|\zeta_1| < |\zeta_1|/4 \tag{84}$$

and, on the other hand, by (75) and (83),

$$|\zeta_2 + \zeta_1 + O(1)| = |f(\zeta_2) - f(\zeta_1)| = \left|\int_\gamma f'(z)dz\right| \leq e^4(200\delta)^{1/2} r < |\zeta_1|/4,$$

which plainly contradicts (84). This proves Theorem 5.

10 Proof of Theorem 6

The following is a special case of an approximation lemma of Edrei and Fuchs [9, 15], and may be proved by estimating $\log|f(z)| - \log|f(0)|$ using the Poisson-Jensen formula.

Lemma K. *There exists a positive absolute constant A with the following property. Let f be meromorphic in the plane with $f(0) = 1$ and with zeros a_μ and poles b_ν, repeated according to multiplicity. Then for large R we have*

$$\log|f(z)| = \sum_{|a_\mu|<2R} \log|1 - z/a_\mu| - \sum_{|b_\nu|<2R} \log|1 - z/b_\nu| + \eta(z, R),$$

in which $|\eta(z, R)| \leq A|z|T(4R, f)/R$ for $|z| \leq R$.

Our proof will be based on combining Lemma K with the following lemma, also used in [27].

Lemma L [36]. *Let $s(t)$ be non-decreasing and integer-valued, continuous from the right, of finite order at most ρ. Suppose that $M > 3$, that $R_0 > 0$ with $s(R_0) > 0$, and that $s(t) = 0$ for $0 \leq t < R_0$. Then there exists a subset E_M of $[R_0, \infty)$, having lower logarithmic density at least $1 - 3/M$, such that for r in E_m we have*

$$s(t)/s(r) \leq (t/r)^{2M(\rho+1)}, \quad t \geq r.$$

Lemma 12. *Let f be transcendental and meromorphic such that (15) holds. Then there exist sequences R_k and S_k tending to infinity, and integers n_k and non-zero constants C_k, such that*

$$f(z) = C_k z^{n_k}(1 + o(1)), \quad R_k/S_k \leq |z| \leq R_k S_k.$$

Proof. We may assume without loss of generality that $f(0) = 1$, and by (15) we may take a sequence (p_k) tending to $+\infty$ such that

$$p_k^{-2} p_{k+1} \to \infty, \quad T(p_k^3, f) = o(\log p_k)^2 \tag{85}$$

as k tends to infinity. Let

$$n(r) = n(r, f) + n(r, 1/f), \quad h(r) = \int_1^r t \, dn(t), \quad m(r) = 2^{n(r)}. \tag{86}$$

Then (85) gives

$$n(r) = o(\log r), \quad m(r) \leq r^{o(1)}, \quad p_k \leq r \leq p_k^2. \tag{87}$$

Let δ be a small positive constant and let $E_0 = \{t \geq 1 : h(t) \geq \delta t\}$. Then by (86),

$$\delta \int_{E_0 \cap [1,r]} dt/t \leq \int_1^r h(t) dt/t^2 \leq O(1) + \int_1^r (1/t) dh(t) = O(1) + \int_1^r dn(t) = o(\log r) \tag{88}$$

for $p_k \leq r \leq p_k^2$. Now define $s(r)$ by

$$s(r) = m(r), \quad p_k \leq r \leq p_k^2,$$

$$s(r) = \left[m(p_k^2) + \frac{(r - p_k^2)}{(p_{k+1} - p_k^2)} (m(p_{k+1}) - m(p_k^2)) \right], \quad p_k^2 < r < p_{k+1}, \tag{89}$$

in which $[x]$ denotes the greatest integer less than or equal to x. Using (85) and (87), we have

$$m(p_k^2) = o(p_k^2) = o(r), \quad m(p_{k+1})/(p_{k+1} - p_k^2) = o(1)$$

for $p_k^2 \leq r \leq p_{k+1}$, and so $s(r) = o(r)$ for all large r. We may thus apply Lemma L, so that if $0 < \delta_1 < 1$ and the positive constant M_1 is chosen large enough, we have

$$s(t)/s(r) \leq (t/r)^{M_1}, \quad t \geq r, \tag{90}$$

for all r lying in a set of lower logarithmic density at least $1 - \delta_1$.

Therefore, provided δ_1 is small enough, the definitions (86) and (89) and the estimates (88) and (90) imply the existence, for all large k, of r_k such that $s(t)/s(r_k) \leq (t/r)^{M_1}$ for all $t \geq r_k$, and

$$p_k^{5/4} \leq r_k \leq p_k^{3/2}, \quad h(r_k) = \int_1^{r_k} t \; dn(t) < \delta r_k,$$

$$n(t) - n(r_k) \leq M_2 \log t/r_k = (M_1/\log 2) \log t/r_k, \quad r_k \leq t \leq p_k^2. \tag{91}$$

Let the zeros and poles of f be a_μ, b_ν, repeated according to multiplicity. Then (91) implies that none of these zeros and poles lies in $\{z : \delta r_k \leq |z| \leq r_k\}$. We may write

$$f(z) = g(z)\Pi_1(1 - z/a_\mu)\Pi_2(1 - z/a_\mu)\Pi_1(1 - z/b_\nu)^{-1}\Pi_2(1 - z/b_\nu)^{-1} \tag{92}$$

in which Π_1 denotes the product over those a_μ (respectively, b_ν) lying in $\{z : 0 < |z| < \delta r_k\}$ and Π_2 denotes the product over those lying in $\{z : r_k < |z| < p_k^2/2\}$. Further, $g(z)$ is analytic and non-zero in $\{z : |z| < p_k^2/2\}$ and we have, by Lemma K, with M_3 a positive absolute constant,

$$|\log |g(z)|| \leq M_3 |z| T(p_k^2, f) p_k^{-2}$$

for $|z| \leq p_k^2/4$ and hence, using (85) and (91),

$$\log |g(z)| = o(1)$$

for $|z| \leq 2r_k$. Thus [19, p. 22] gives $g'(z)/g(z) = o(1/r_k)$ for $|z| \leq r_k$ and so, since $g(0) = f(0) = 1$,

$$g(z) = 1 + o(1), \quad |z| \leq r_k. \tag{93}$$

It remains only to estimate the product terms in (92) and this we do in the following standard way. First, we write

$$\Pi_1(1 - z/a_\mu) = \Pi_1(-z/a_\mu)\Pi_1(1 - a_\mu/z)$$

and obtain, for $\delta^{3/4} r_k \leq |z| \leq \delta^{1/4} r_k$,

$$|\Pi_1(1 - a_\mu/z) - 1| \leq \exp(\Sigma_1 |a_\mu/z|) - 1 \leq \exp(h(r_k)/\delta^{3/4} r_k) - 1 \leq \exp(\delta^{1/4}) - 1,$$

using (91). Further, again for $\delta^{3/4} r_k \leq |z| \leq \delta^{1/4} r_k$, we have

$$|\Pi_2(1 - z/a_\mu) - 1| \leq \exp(|z| \int_{r_k}^{p_k^2/2} (1/t) dn(t, 1/f)) - 1.$$

But, for these z, using (87) and (91),

$$\begin{aligned}
|z| \int_{r_k}^{p_k^2/2} (1/t) dn(t, 1/f) &\leq 2|z| n(p_k^2/2, 1/f)/p_k^2 + |z| \int_{r_k}^{p_k^2/2} (n(t, 1/f) - n(r_k, 1/f)) dt/t^2 \\
&\leq o(1) + |z| M_2 \int_{r_k}^{\infty} (\log t/r_k) \; dt/t^2 \\
&\leq o(1) + |z| M_2/r_k \leq o(1) + M_2 \delta^{1/4}.
\end{aligned}$$

Applying similar estimates to the products over poles and using (93), Lemma 12 is proved.

Suppose that $f(z)$ satisfies the hypotheses of Theorem 6, but has only finitely many critical values. Let the sequences R_k, S_k, C_k, n_k be as in Lemma 12.

We assert first the existence of a value w in the extended complex plane and a positive sequence t_k tending to infinity such that $f(z)$ tends to w as z tends to infinity in the union of the circles $|z| = t_k$. This is obvious, with $t_k = R_k$, if $n_k = 0$ for infinitely many k. In the case where $n_k \neq 0$ for all large k the assertion again follows immediately unless there are positive constants c_j such that

$$c_1 \leq |C_k| R_k^{n_k} \leq c_2$$

for all large k. In this case, however, the assertion follows at once with $w = 0$ or $w = \infty$ and with $\log t_k = \log R_k \pm (1/2) \log S_k$.

Our assertion having been established, we may assume without loss of generality that $w = \infty$, and it is immediate that $f(z)$ cannot have a finite asymptotic value. We proceed to a contradiction using a method simpler than that of [27]. Since we are assuming that f has only finitely many critical values, it follows that there exists a positive constant T such that f has no asymptotic or critical value in $\{u : T/2 < |u| < \infty\}$, so that by [38, Chapter XI] all components D_ν of the set $\{z : |f(z)| > T\}$ are simply connected. But, for large k, the circle $|z| = t_k$ lies in such a component D_ν and this is obviously a contradiction. Theorem 6 is proved.

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