On the zeros of $f^{(k)}/f$

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Abstract

Suppose that k is a positive integer and that f is meromorphic of finite order in the plane, such that $g = f^{(k)}/f$ is transcendental. If g has order less than $1/2$ then g must have infinitely many zeros, and the same conclusion holds if f has only finitely many poles and $\liminf_{r\to\infty} T(r,f)/r = 0$.

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1 INTRODUCTION

As part of an investigation of the equilibrium points of potentials $u(z) = \sum \log|z-a|$, the following sharp theorem was proved in [3]. The notation is that of [7].

Theorem A. *Suppose that f is transcendental and meromorphic with $T(r,f) = o(r^{1/2})$ as $r \to \infty$, or entire with $T(r,f) = o(r)$ as $r \to \infty$. Then f'/f has infinitely many zeros.*

Related results may be found in [2, 3, 4, 5, 10]. We consider here the analogous problem in which f'/f is replaced by $f^{(k)}/f$, with $k \geq 2$. Here the case of entire f goes through almost exactly as for $k = 1$. However, for meromorphic f, the problem seems rather more complicated, for the following reason. If $T(r,f) = o(r^{1/2})$ and f'/f is transcendental with only finitely many zeros, it is possible [2, 3, 4] to find a path Γ tending to infinity and a sequence of circles $|z| = r_n, r_n \to \infty$, on the union S of which $f(z)$ tends to a finite, non-zero value, c say, as z tends to infinity. Further, if R and ε are positive with ε and $1/R$ small enough, then the set $\{z \in S : |z| > R\}$ lies in a component U of the set $\{z : |f(z) - c| < \varepsilon\}$ such that U contains no multiple points of f. This in turn implies [2, 3, 12] that U must be simply connected, which is clearly impossible. For $k \geq 2$ and $f^{(k)}/f$ transcendental and zero-free, similar arguments give a finite asymptotic value c of $f^{(k-1)}$ but it is not so immediate that we get a component of $\{z : |f^{(k-1)}(z) - c| < \varepsilon\}$ which contains no multiple points of $f^{(k-1)}$. Thus we use instead a local argument, based on an application of the Riemann-Hurwitz formula [13].

Theorem 1. *Suppose that k is a positive integer and that f is meromorphic of finite order in the plane such that $f^{(k)}/f$ is transcendental. If $f^{(k)}/f$ has order less than $1/2$ then*

$f^{(k)}/f$ has infinitely many zeros. The same conclusion holds if f has only finitely many poles and $\liminf_{r\to\infty} T(r,f)/r = 0$.

The paper [3] contains examples, for every $\rho \geq 1/2$, of meromorphic f of order ρ such that f'/f has no zeros: for $\rho = 1/2$ we may take $f(z) = \tan^2 \sqrt{z}$. Further, for each positive integer k there is a solution F of the equation $z^{k-1}w^{(k)} = w$, entire of order $1/k$, so that the assumption in Theorem 1 that $f^{(k)}/f$ is transcendental is not redundant.

2 LEMMAS NEEDED FOR THEOREM 1

Lemma A. *Let d_1 be a positive constant and let G be transcendental and meromorphic with $T(r,G) = O(r^{d_1})$ as $r \to \infty$. Then there are positive constants c_j with the following properties. If $r > c_1$ and $c_2 < S < \exp(r^{c_3})$ then there exist uncountably many R with $S < R < 2S$ such that the length $L(r,R,G)$ of the level curves $|G(z)| = R$ lying in $|z| < r$ satisfies $L(r,R,G) \leq r^{c_4}$.*

In addition, there exist a constant $d_2 > 0$, depending only on d_1, and arbitrarily large positive constants T, such that G has no critical values on $|w| = T$, while $L(r,T,G) = O(r^{d_2})$ as $r \to \infty$.

The first assertion of Lemma A is a standard application of the length-area inequality as used in [15] (see also [2, 4]). The second assertion is proved in [10].

The next lemma is a well known consequence of Tsuji's estimate for harmonic measure [14], $B(r,u)$ being defined by $B(r,u) = \sup\{u(z) : |z| = r\}$.

Lemma B. *Let U_1, U_2 be non-constant subharmonic functions in the plane, and let D_1, D_2 be disjoint, unbounded domains such that $U_j \leq 0$ on ∂D_j and $U_j(z_j) > 0$ for at least one point z_j in D_j. Then, for all large r,*

$$\log B(r,U_1) + \log B(r,U_2) \geq 2\log r - O(1).$$

We need the following special case of the Riemann-Hurwitz formula [13]: if D and G are bounded domains of connectivity m,n respectively, and if F is a function analytic on the closure of D, mapping D onto G and ∂D onto ∂G, with no multiple point in D, then $m - 2 = p(n-2)$, in which p is the degree of the mapping F.

3 PROOF OF THEOREM 1

Suppose that f is meromorphic of finite order in the plane, such that $f^{(k)}/f$ is transcendental but has only finitely many zeros, for some positive integer k. Set

$$h(z) = f(z)/f^{(k)}(z), \quad f_1(z) = f(z)z^{1-k}, \quad h_1(z) = h(z)z^{-N}, \quad N = d_2 + 2k + 1, \quad (1)$$

in which d_2 is the constant arising from applying Lemma A with $d_1 - 1$ the order of h. We note that h has only finitely many poles. By a result of Lewis, Rossi and Weitsman [11], there is a simple path Γ_0 tending to infinity in $\{z : |z| \geq 1\}$, such that

$$\frac{\log |h(z)|}{\log |z|} \to +\infty \tag{2}$$

as z tends to infinity on Γ_0, and

$$\int_{\Gamma_0} \exp(-\delta \log |h(t)|)|dt| < \infty \tag{3}$$

for every positive constant δ.

By moving the starting point z_0 of Γ_0, if necessary, we may assume that

$$\int_{\Gamma_0} k|t|^{2k}|h(t)|^{-1}|dt| \leq \int_{\Gamma_0} |h(t)|^{-1/2}|dt| < \infty, \tag{4}$$

using (2) and (3). We choose $t_1 > 2$, so large that h has no poles in $\{z : |z| \geq t_1\}$, and we apply Lemma A to obtain $T_1 > M(t_1, h_1)$, such that $L(r, T_1, h_1) = O(r^{d_2})$ as $r \to \infty$, while h_1 has no critical values on $|w| = T_1$. We may assume that the path Γ_0 lies in a component C of the set $\{z : |h_1(z)| > T_1\}$. Partitioning the boundary ∂C of C into its intersections with the annuli $\{z : 2^{m-1} \leq |z| < 2^m\}$, we have, by the choice of T_1,

$$\int_{\partial C} |t|^{2k}|h(t)|^{-1}|dt| \leq \sum_{m=1}^{\infty} c2^{md_2}2^{2km}2^{(m-1)(-2k-1-d_2)} < \infty, \tag{5}$$

using c henceforth to denote a positive constant, not necessarily the same at each occurrence. Now any point ζ in the closure of C may be joined to z_0 by a path μ_ζ consisting of part of the circle $|z| = |z_0|$, and part of the ray $\arg z = \arg \zeta$. We may replace any part of μ_ζ which leaves the closure of C by an arc of ∂C, and thus we have $|h_1(z)| \geq T_1$ on μ_ζ, and so, using (5),

$$\int_{\mu_\zeta} |t|^{2k}|h(t)|^{-1}|dt| \leq c + c\int_{|z_0|}^{\infty} s^{2k-N} ds \leq c, \tag{6}$$

with the constants independent of ζ.

We have, near z_0,

$$f(z) - P_1(z) = \int_{z_0}^{z} \frac{(z-t)^{k-1}}{(k-1)!} f(t)h(t)^{-1}dt, \tag{7}$$

in which P_1 is a polynomial of degree at most $k - 1$. Expanding out the $(z - t)^{k-1}$ term, we see that the integral in (7) may be analytically continued throughout the closure of C, and (7) continues to hold, since the left hand side is single-valued.

The function f may now be estimated on C, using a standard method (see, for example, [9]). For ζ in the closure of C, we parametrize μ_ζ with respect to arc length s. It follows from (7) that

$$|f_1(\mu_\zeta(s))| \leq c + \int_0^s \left| \frac{(\mu_\zeta(s) - \mu_\zeta(t))^{k-1}\mu_\zeta(t)^{k-1} f_1(\mu_\zeta(t))}{(k-1)!\mu_\zeta(s)^{k-1} h(\mu_\zeta(t))} \right| dt \leq$$

$$\leq W(s) = c + \int_0^s k|\mu_\zeta(t)|^{2k-2}|f_1(\mu_\zeta(t))h(\mu_\zeta(t))^{-1}|dt. \tag{8}$$

Thus

$$dW/ds \leq k|\mu_\zeta(s)|^{2k-2}|h(\mu_\zeta(s))|^{-1}W(s)$$

and so

$$W(s) \leq c\exp\left(\int_0^s k|\mu_\zeta(t)|^{2k-2}|h(\mu_\zeta(t))|^{-1}dt\right) \leq c,$$

using (6), which gives

$$f(z) = O(|z|^{k-1}) \tag{9}$$

for z in C. Further, differentiating (7) and using (6) and (9), we see that $f^{(k-1)}$ is bounded on C.

The assertion of the theorem, in the case where f has only finitely many poles, now follows from a standard argument. Defining a subharmonic function U_1, identically zero outside C, and equal to $\log|h_1(z)/T_1|$ on C, the Poisson-Jensen formula gives, for $j = 1$,

$$B(r, U_j) = O(m(2r, U_j)) = O(T(2r, f)) \tag{10}$$

as $r \to \infty$. But we can choose t_2, large and positive, such that f has no poles in $\{z : |z| \geq t_2\}$, as well as $T_2 > M(t_2, f^{(k-1)})$, so large that an unbounded component C' of the set $\{z : |f^{(k-1)}(z)| > T_2\}$ does not meet C. Defining $U_2(z)$ to be $\log|f^{(k-1)}(z)/T_2|$ on C' and 0 elsewhere, we have (10), for $j = 2$, and a contradiction arises on applying Lemma B.

We turn now to the case where h has order less than 1/2, and first note that, by (4) and (9), the integral

$$\int_{\Gamma_0} f(t)h(t)^{-1}dt$$

converges and, consequently, there is a constant α such that

$$f^{(k-1)}(z) = \alpha - \int_z^\infty f(t)h(t)^{-1}dt = \alpha + o(1) \tag{11}$$

as z tends to infinity on Γ_0.

The path Γ_0 and the component C above are not quite adequate for our subsequent requirements and we replace them as follows. By the well known $\cos\pi\rho$ theorem [1, 6] the maximum $M(r, h)$ and minimum $m_0(r, h)$ of $|h(z)|$ on $|z| = r$ are such that $\log m_0(r, h) > c\log M(r, h)$ on a subset E_1 of $(1, +\infty)$ of lower logarithmic density at least c. If the positive constant c' is large enough then, for all large r, the interval $[r, r^{c'}]$ meets E_1. We thus choose a sequence (r_n) such that

$$(r_n)^{1+c} < r_{n+1} < (r_n)^{(1+c)^2} \tag{12}$$

and

$$m_0(r_n, h)^{1+c} > M(r_n, h). \tag{13}$$

We may also assume that

$$\left|\frac{f^{(k)}(z)}{f^{(k-1)}(z) - \alpha}\right| \leq (r_n)^c, \quad |z| = r_n, \tag{14}$$

as may be ensured by first deleting from E_1 a set of finite measure. Provided the constant c in (12) is chosen large enough, the convexity of $\log M(r,h)$ as a function of $\log r$ implies that

$$m_0(r_{n+1},h) > M(r_n,h)^{1+c}, \tag{15}$$

and obviously

$$\frac{\log m_0(r_n,h)}{\log r_n} \to +\infty, \quad n \to +\infty. \tag{16}$$

We now choose M_n satisfying

$$M(r_n,h)/4 \leq M_n \leq M(r_n,h)/2 \tag{17}$$

and such that h has no critical values on $|w| = M_n$, while the length $L(r, M_n, h)$ of the level curves $|h(z)| = M_n$ lying in $|z| < r$ satisfies

$$L(r_{n+2}, M_n, h) \leq (r_n)^c. \tag{18}$$

Here we have used Lemma A, (12), and the fact that $\log M_n = O(T(r_n,h))$.

Now, some point z^* on $|z| = r_n$ lies in a component C_1 of the set $\{z : |h(z)| > M_n\}$, and C_1 must meet $|z| = r_{n+1}$. We may therefore join z^* to $|z| = r_{n+1}$ by a path τ_n consisting of part of the circles $|z| = r_n, |z| = r_{n+1}$, part of a straight line through the origin, and part of the boundary of C_1, such that $|h(z)| \geq M_n$ on τ_n. This path τ_n has a simple sub-path γ_n which lies in $\{z : r_n \leq |z| \leq r_{n+1}\}$ and joins $|z| = r_n$ to $|z| = r_{n+1}$, and γ_n has length at most $O(r_n)^c$, using (12) and (18).

We now form a path Γ_1 tending to infinity, not simple, such that $\{\Gamma_1\}$ consists of the union of the circles $|z| = r_n$ and the paths γ_n, for $n \geq n_0$, say. The part of Γ_1 lying in $|z| \leq r_n$ has length at most $O(r_n)^c$. Further, each z lying on Γ_1 with $r_n \leq |z| < r_{n+1}$ can be joined to infinity by a sub-path $\Lambda_1(z)$ of Γ_1 contained in the union of the circles $|t| = r_m$ and the paths γ_m, for $m \geq n$. By construction of Γ_1 there is a positive constant d such that we have, using (12) and (16),

$$|z| \leq |t|^d \text{ for } t \in \Lambda_1(z), \quad \int_{\Gamma_1} |t|^{(2+d)k} |h(t)|^{-1} |dt| \leq \sum_{n=n_0}^{\infty} (r_n)^c m_0(r_n,h)^{-1} < 1/2, \tag{19}$$

provided n_0 was chosen large enough. Choose u_0 on $|z| = r_{n_0}$ and a polynomial P_2 such that

$$P_2^{(j)}(u_0) = f^{(j)}(u_0), \quad j = 0, \ldots, k-1.$$

Then we have

$$f(z) - P_2(z) = \int_{u_0}^{z} \frac{(z-t)^{k-1}}{(k-1)!} f(t) h(t)^{-1} dt \tag{20}$$

near u_0, and this remains true for all z on Γ_1, by analytic continuation, using the fact that the left hand side is single-valued. The same argument as used on C now shows that (9) holds on Γ_1.

Hence the function

$$\int_z^{\infty} (z-t)^{k-1} f(t) h(t)^{-1} dt,$$

in which the integration is eventually along Γ_1, admits analytic continuation on a neighbourhood of Γ_1 and there is a polynomial P, of degree at most $k-1$, such that

$$f^{(j)}(z) - P^{(j)}(z) = -\int_z^\infty \frac{(z-t)^{k-1-j}}{(k-1-j)!} f(t)h(t)^{-1}dt, \quad j = 0, \ldots, k-1, \tag{21}$$

on a neighbourhood of u_0 and hence on all of Γ_1.

We estimate $f^{(j)}(z) - P^{(j)}(z)$, for z on Γ_1. Let σ_n be the union of the circle $|z| = r_n$ and the path γ_n from $|z| = r_n$ to $|z| = r_{n+1}$ and let z lie on σ_n. Then for t on $\sigma_m, m \geq n$, we have

$$|z-t| \leq (r_m)^c, \quad |f(t)| \leq (r_m)^c, \quad |h(t)| \geq m_0(r_m, h)/4.$$

Thus, for $j = 0, \ldots, k-1$, (21) gives

$$|f^{(j)}(z) - P^{(j)}(z)| \leq \sum_{m=n}^{\infty} (r_m)^c m_0(r_m, h)^{-1} \leq 2(r_n)^c m_0(r_n, h)^{-1} = S_n, \quad z \in \sigma_n, \tag{22}$$

using (12), (13), (15) and (16). Recalling (11), we have $P^{(k-1)}(z) \equiv \alpha$.

We assert that $P(z) \not\equiv 0$. To see this, assume the contrary. Then it follows from (22) that $f(z)$ tends to 0 as z tends to infinity on Γ_1. We may choose arbitrarily small positive ε and z on Γ_1 with $|f(z)| = \varepsilon$ and $|f(t)| \leq \varepsilon$ on the subpath $\Lambda_1(z)$ of Γ_1 joining z to infinity. Applying (19) and (21) gives $|f(z)| \leq \varepsilon/2$, a contradiction.

By (14) and (22) and the fact that $P(z) \not\equiv 0$ we have, for large n,

$$|f^{(k-1)}(z) - \alpha| \geq (r_n)^{-c} M(r_n, h)^{-1} = s_n, \quad |z| = r_n. \tag{23}$$

We now choose V_n with

$$2S_n < V_n < 4S_n, \quad L(r_{n+4}, V_n, f^{(k-1)} - \alpha) \leq (r_n)^c, \tag{24}$$

and such that $f^{(k-1)} - \alpha$ has no critical values w with $|w| = V_n$. Here we apply Lemma A to the function $1/(f^{(k-1)} - \alpha)$. We note that, by (12), (15) and (16),

$$s_n \leq S_n < V_n < (1/8)s_{n-1}, \quad \frac{\log 1/V_n}{\log r_n} \to \infty, \quad n \to \infty. \tag{25}$$

It follows from (22) that we can choose arbitrarily large n such that the function $f^{(k-1)} - \alpha$ has at least one pole in $\{z : r_n < |z| < r_{n+1}\}$. On the union η_n of the circles $|z| = r_n$, $|z| = r_{n+1}$ and the path γ_n, we have $|f^{(k-1)}(z) - \alpha| \leq S_n$, and η_n lies in a component E_n of the set $\{z : |z| < r_{n+3}, |f^{(k-1)}(z) - \alpha| < V_n\}$. We assert that f has no zeros in E_n. If $k = 1$, this is obvious, since $\alpha \equiv P^{(k-1)}(z)$ and $P(z) \not\equiv 0$. If $k \geq 2$, we choose a point z_3 on $|z| = r_n$. Now, any point z_4 in the closure of E_n can be joined to z_3 by a path γ^*, of length $O(r_n)^c$, on which $|f^{(k-1)}(z) - \alpha| \leq V_n$. The estimate (22) at z_3 and the formula

$$f(z) - P(z) = \sum_{j=0}^{k-2} (f^{(j)}(z_3) - P^{(j)}(z_3))(z - z_3)^j/j! + \int_{z_3}^{z} \frac{(z-t)^{k-2}}{(k-2)!}(f^{(k-1)}(t) - \alpha)dt$$

give $|f(z) - P(z)| \leq (r_n)^c V_n = o(1)$ on E_n, using (25), and our assertion is proved. Since $|f^{(k-1)}(z) - \alpha| \geq s_{n-1} > V_n$ on $|z| = r_{n-1}$, the component E_n lies in $\{z : |z| > r_{n-1}\}$ and so $f^{(k)}(z) \neq 0$ on E_n.

Suppose now that F_n is a component of the set $\{z : |f^{(k-1)}(z) - \alpha| < (1/2)s_{n+1}\}$ which meets γ_n. Then F_n does not meet the circles $|z| = r_n, |z| = r_{n+1}$, by (23). Thus F_n is contained in E_n and $f^{(k)}(z) \neq 0$ on F_n, so that F_n is simply connected, by the Riemann-Hurwitz formula. We can therefore replace each arc of γ_n which meets F_n by an arc of the boundary of F_n, and doing this gives us a path λ_n joining $|z| = r_n$ to $|z| = r_{n+1}$ and lying in $\{z : r_n \leq |z| \leq r_{n+1}\}$, such that we have $(1/2)s_{n+1} \leq |f^{(k-1)}(z) - \alpha| < V_n$ on the union ν_n of λ_n and the circles $|z| = r_n, |z| = r_{n+1}$. On $|z| = r_{n+2}$ we have $|f^{(k-1)}(z) - \alpha| \leq S_{n+2} < (1/8)s_{n+1}$, while on $|z| = r_{n-1}$ we have $|f^{(k-1)}(z) - \alpha| \geq s_{n-1} > V_n$. Thus ν_n lies in a component D_n of the set $\{z : (1/4)s_{n+1} < |f^{(k-1)}(z) - \alpha| < V_n\}$, and the closure of D_n cannot meet either of the circles $|z| = r_{n-1}, |z| = r_{n+2}$. Since $f^{(k-1)} - \alpha$ has a pole in $\{z : r_n < |z| < r_{n+1}\}$, this component D_n must be at least triply connected. However, since the circle $|z| = r_n$ lies in both E_n and D_n, it follows that D_n must be contained in E_n, so that $f^{(k)}(z) \neq 0$ on D_n and, by the Riemann-Hurwitz formula, D_n must be doubly connected. We have reached a contradiction and the theorem is proved.

We remark that the proof above for order less than 1/2 does not seem to work in the case $T(r, f^{(k)}/f) = o(r^{1/2})$ because of the need to make $m_0(r_{n+1}, h)$ large compared to $M(r_n, h)$, while keeping r_{n+1} not too large compared to r_n.

References

[1] P.D. Barry, On a theorem of Besicovitch, Quart. J. of Math. Oxford **(2) 14** (1963), 293-302.

[2] W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, Rev. Mat. Iberoamericana **11** (1995), 355-373.

[3] J. Clunie, A. Eremenko and J. Rossi, On equilibrium points of logarithmic and Newtonian potentials, J. London Math. Soc. **(2) 47** (1993), 309-320.

[4] A. Eremenko, J.K. Langley and J. Rossi, On the zeros of meromorphic functions of the form $\sum_{k=1}^{\infty} \frac{a_k}{z-z_k}$, J. d'Analyse Math. **62** (1994), 271-286.

[5] A.A. Gol'dberg and M.E. Korenkov, The defect at zero of the logarithmic derivative of an entire function, Teor. Funktsii Funktsional. Anal. i Prilozhen. **34** (1980), 41-46.

[6] A.A. Gol'dberg and O.P. Sokolovskaya, Some relations for meromorphic functions of order or lower order less than one, Izv. Vyssh. Uchebn. Zaved. Mat. **31** (1987), 26-31. Translation: Soviet Math. (Izv. VUZ) **31** (1987), 29-35.

[7] W.K. Hayman, Meromorphic functions, Oxford at the Clarendon Press, 1964.

[8] W.K. Hayman, Subharmonic functions Vol. 2, Academic Press, London, 1989.

[9] J.K. Langley, Proof of a conjecture of Hayman concerning f and f'', J. London Math. Soc. **(2) 48** (1993), 500-514.

[10] J.K. Langley and D.F. Shea, On multiple points of meromorphic functions, to appear, J. London Math. Soc.

[11] J. Lewis, J. Rossi and A. Weitsman, On the growth of subharmonic functions along paths, Ark. Mat. **22** (1983), 104-114.

[12] R. Nevanlinna, Eindeutige analytische Funktionen, Springer, Berlin, 1953.

[13] N. Steinmetz, Rational iteration, de Gruyter Studies in Mathematics 16, Walter de Gruyter, Berlin/New York, 1993.

[14] M. Tsuji, Potential theory in modern function theory, Maruzen, Tokyo, 1959.

[15] A. Weitsman, A theorem on Nevanlinna deficiencies, Acta Math. **128** (1972), 41-52.