# Value distribution of differences of meromorphic functions

J.K. Langley *

January 6, 2009

**Abstract**

Let $f$ be a function transcendental and meromorphic in the plane. Results are proved concerning the existence of zeros of the $n$'th forward difference $\Delta^n f$ and the divided difference $\Delta^n f/f$. **MSC 2000:** 30D35.

# 1 Introduction

Let the function $f$ be transcendental and meromorphic in the plane. The forward differences $\Delta^n f$ are defined by [25, p.52]

$$\Delta f(z) = f(z+1) - f(z), \quad \Delta^{n+1} f(z) = \Delta^n f(z+1) - \Delta^n f(z), \quad n = 1, 2, \ldots. \tag{1.1}$$

This paper continues the investigations of [4] into the zeros of the forward differences $\Delta^n f$ as defined in (1.1) and the divided differences $\Delta^n f/f$. The work in [4] reflects in part the considerable attention given recently to meromorphic solutions in the plane of difference and functional equations [1, 5, 8, 9, 13, 17], but the results from [4] may also be viewed as discrete analogues of the following sharp theorem [6, 16], which uses notation from [10].

**Theorem 1.1 ([6, 16])** *Let $f$ be transcendental and meromorphic in the plane with*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0.$$

*Then $f'$ has infinitely many zeros.*

The following result was proved in [4] using Wiman-Valiron theory [11].

**Theorem 1.2 ([4])** *Let $n \in \mathbb{N}$ and let $f$ be a transcendental entire function of order $\rho < \frac{1}{2}$, and set*

$$G_n(z) = \frac{\Delta^n f(z)}{f(z)}. \tag{1.2}$$

*If $G_n$ is transcendental then $G_n$ has infinitely many zeros. In particular if $f$ has order less than $\min\left\{\frac{1}{n}, \frac{1}{2}\right\}$ then $G_n$ is transcendental and has infinitely many zeros.*

*Research supported by Engineering and Physical Sciences Research Council grant EP/D065321/1

Note that if $f$ is an entire function of order less than $1/2$ for which $G_n$ fails to be transcendental for some $n \geq 2$ then $f$ satisfies a homogeneous linear difference equation with rational coefficients and the growth of such solutions was determined in [17]. For the first divided difference Theorem 1.2 was extended slightly beyond $\rho = 1/2$ in [4].

**Theorem 1.3 ([4])** *There exists $\delta_0 \in (0, 1/2)$ with the following property. Let $f$ be a transcendental entire function with order $\rho(f) < 1/2 + \delta_0$. Then*

$$G(z) = G_1(z) = \frac{\Delta f(z)}{f(z)} = \frac{f(z+1) - f(z)}{f(z)} \tag{1.3}$$

*has infinitely many zeros.*

The constant $\delta_0$ in Theorem 1.3 is extremely small, but it was conjectured in [4] that the conclusion of Theorem 1.3 holds for all entire $f$ with $\rho(f) < 1$. The first result of the present paper extends Theorem 1.3 beyond order $1/2$ for higher divided differences, and broadens the applicability of Theorem 1.2 to meromorphic functions with few poles, even for order $1/2$.

**Theorem 1.4** *Let $n \in \mathbb{N}$. Let $f$ be transcendental and meromorphic of order $\rho < 1$ in the plane and assume that $G_n$ as defined by (1.2) is transcendental.*
*(i) If $G_n$ has lower order $\mu < \alpha < 1/2$, which holds in particular if $\rho < 1/2$, then*

$$\delta(0, G_n) \leq 1 - \cos \pi\alpha \quad \textit{or} \quad \delta(\infty, f) \leq \frac{\mu}{\alpha}.$$

*(ii) If $\rho = 1/2$ then either $G_n$ has infinitely many zeros or $\delta(\infty, f) < 1$.*
*(iii) If $f$ is entire and $\rho < 1/2 + \delta_0$, then $G_n$ has infinitely many zeros: here $\delta_0$ is a small positive absolute constant.*

For meromorphic functions in general the following theorem was proved in [4], and addressed a question which represents a natural discrete analogue of Theorem 1.1: if $f$ is transcendental with $\rho(f) < 1$ must $\Delta f$ has infinitely many zeros?

**Theorem 1.5 ([4])** *Let $f$ be a function transcendental and meromorphic in the plane of lower order $\lambda(f) < 1$. Let $c \in \mathbb{C} \setminus \{0\}$ be such that at most finitely many poles $z_j, z_k$ of $f$ satisfy $z_j - z_k = c$. Then $g(z) = f(z+c) - f(z)$ has infinitely many zeros.*

Clearly all but countably many $c \in \mathbb{C}$ satisfy the hypotheses of Theorem 1.5, but the following construction from [4] showed that Theorem 1.5 fails without the hypothesis on $c$, even for lower order $0$, and that if the answer to the above question for meromorphic functions is affirmative, then in contrast to Theorem 1.1 it depends on order and not lower order.

**Theorem 1.6 ([4])** *Let $\phi(r)$ be a positive non-decreasing function defined on $[1, \infty)$ which satisfies $\lim_{r\to\infty} \phi(r) = \infty$. Then there exists a function $f$ transcendental and meromorphic in the plane such that $g(z) = \Delta f(z)$ has only one zero and*

$$\limsup_{r\to\infty} \frac{T(r,f)}{r} < \infty, \quad \liminf_{r\to\infty} \frac{T(r,f)}{\phi(r)\log r} < \infty, \quad \limsup_{r\to\infty} \frac{T(r,g)}{\phi(r)\log r} < \infty.$$

The final theorem from [4] showed that for transcendental meromorphic functions satisfying the very strong growth restriction $T(r,f) = O(\log r)^2$ as $r \to \infty$, either the first difference or the first divided difference has infinitely many zeros. The proof of this result depended on asymptotic properties of such functions with deficient poles [2], but this reliance is dispensed with in the following substantial improvement.

**Theorem 1.7** *Let $f$ be a transcendental meromorphic function in the plane, of order less than $1/6$, and define $G$ by (1.3). Then at least one of $G$ and $\Delta f$ has infinitely many zeros.*

This research was carried out during a visit to the University of Joensuu in September 2007. The author thanks the Department of Mathematics for its hospitality, and Ilpo Laine and Risto Korhonen for very interesting discussions.

## 2 Preliminaries for Theorem 1.4

A key role for Theorem 1.4(iii) will be played by the following result of Miles and Rossi [23].

**Lemma 2.1 ([23])** *Let $f$ be a transcendental entire function of order $\rho(f) \leq \rho < \infty$. Let $0 < \gamma < 1$, and for $r > 0$ let*

$$U_r = \left\{ \theta \in [0, 2\pi] : \left| \frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} \right| \geq \gamma n(r, 1/f) \right\}. \tag{2.1}$$

*Let $M > 3$. Then there exists a set $Q_M \subseteq [1, \infty)$ of lower logarithmic density*

$$\underline{\mathrm{logdens}}\, Q_M = \liminf_{r \to \infty} \left( \frac{1}{\log r} \int_{[1,r] \cap Q_M} \frac{dt}{t} \right) \geq 1 - \frac{3}{M}, \tag{2.2}$$

*such that*

$$m(U_r) > \left( \frac{1 - \gamma}{7M(\rho + 1)} \right)^2 \quad \text{for} \quad r \in Q_M, \tag{2.3}$$

*in which $m(U_r)$ denotes the Lebesgue measure of $U_r$ .*

□

The next lemma is a version of the celebrated $\cos \pi\lambda$ theorem [12, Chapter 6].

**Lemma 2.2 ([7])** *Suppose that $g$ is transcendental and meromorphic in the plane, of lower order $\mu < \alpha < 1$, and define $L(r, g) = \min\{|g(z)| : |z| = r\}$ and*

$$Y_1 = \{r > 1 : \log L(r, g) > \gamma(\cos \pi\alpha + \delta(\infty, g) - 1)T(r, g)\}, \quad \gamma = \frac{\pi\alpha}{\sin \pi\alpha}.$$

*Then $Y_1$ has upper logarithmic density at least $1 - \mu/\alpha$.*

□

**Lemma 2.3 ([4])** *Let $H$ be a transcendental entire function of order $\rho_1 < \infty$. For large $r > 0$ define $r\theta(r)$ to be the length of the longest arc of the circle $S(0, r)$ of centre $0$ and radius $r$ on which $|H(z)| > 1$, with $\theta(r) = 2\pi$ if $|H(z)| > 1$ on all of $S(0, r)$, that is, $L(r, H) > 1$. Then at least one of the following is true:*
*(a) there exists a set $F \subseteq [1, \infty)$ of positive upper logarithmic density such that $L(r, H) > 1$ for $r \in F$;*
*(b) for each $\tau \in (0, 1)$ the set*

$$F_\tau = \{r \geq 1 : \theta(r) > 2\pi(1 - \tau)\} \tag{2.4}$$

*satisfies*

$$\underline{\mathrm{logdens}}\, F_\tau \geq \frac{1 - 2\rho_1(1 - \tau)}{\tau}. \tag{2.5}$$

If $H$ has lower order less than $1/2$, which of course is true if $\rho_1 < 1/2$, then case (a) always holds [3]. Moreover if $\rho_1 = 1/2$ then $\theta(r) \to 2\pi$ on a set of positive upper logarithmic density. We outline the standard argument for this assertion, which is obvious if case (a) applies, so assume that $H$ satisfies case (b). Then the right-hand side of (2.5) is $1$, and so for each $n \in \mathbb{N}$ the set

$$P_n = \{r \geq 1 : 2\pi - \theta(r) \geq 1/n\}$$

has logarithmic density $0$ using (2.4). Hence we may choose a sequence $(s_n)$ increasing to infinity such that

$$\int_{[1,r]\cap P_n} \frac{dt}{t} \leq \frac{\log r}{n}$$

for $r \geq s_n$. Let $P$ be the union of the sets $P_n \cap [s_n, s_{n+1})$. Then $\theta(r) \to 2\pi$ as $r$ tends to infinity outside $P$. For large $r$ choose $n$ with $s_n \leq r < s_{n+1}$. Then

$$\int_{[1,r]\cap P} \frac{dt}{t} \leq \int_{[1,r]\cap P_n} \frac{dt}{t} \leq \frac{\log r}{n}$$

and so $P$ has logarithmic density $0$. $\square$

**Lemma 2.4 ([4])** *Let $n \in \mathbb{N}$. Let $f$ be transcendental and meromorphic of order less than $1$ in the plane. Then there exists a set $X_0 \subseteq [1, \infty)$ of finite logarithmic measure such that*

$$G_n(z) = \frac{\Delta^n f(z)}{f(z)} \sim \frac{f^{(n)}(z)}{f(z)} = o(1) \quad \textit{as } z \to \infty \textit{ with } |z| \notin X_0. \tag{2.6}$$

$\square$

The proof of the following lemma is related to that of Theorem 4 in [18], but the present approach is somewhat simpler.

**Lemma 2.5** *Let $f$ be transcendental and meromorphic in the plane and let $n \in \mathbb{N}$. Let $c > 0$ and let $E$ be an unbounded subset of $[1, \infty)$ with the following property. For each $r \in E$ there exists a compact arc $\Omega_r$ of the circle $S(0, r)$, of angular measure at least $c$, such that*

$$\lim_{r\to\infty, r\in E} r^{2n} M(\Omega_r, f^{(n)}/f) = 0, \quad \textit{where} \quad M(\Omega_r, g) = \max\{|g(z)| : z \in \Omega_r\}. \tag{2.7}$$

*Let $\phi(r)$ be a positive function tending to infinity with $\phi(r) = o(\log r)$ as $r$ tends to infinity. Then for all sufficiently large $r \in E$ we have*

$$\left|\frac{zf'(z)}{f(z)}\right| \leq n\phi(r) \tag{2.8}$$

*for all $z \in \Omega_r$ outside a set of discs having sum of radii at most $(n-1)r/\phi(r)$.*

*Proof.* There is nothing to prove if $n = 1$ so assume that $n \geq 2$. Let $r \in E$ be large and choose $z_r \in \Omega_r$ with

$$|f(z_r)| = M_r = M(\Omega_r, f). \tag{2.9}$$

Now Taylor's formula gives a polynomial $P$ depending on $r$ and of degree at most $n-1$ such that, for $z \in \Omega_r$,

$$f(z) = P(z) + \int_{z_r}^{z} \frac{(z-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt, \quad f'(z) = P'(z) + \int_{z_r}^{z} \frac{(z-t)^{n-2}}{(n-2)!} f^{(n)}(t)\,dt.$$

It follows from (2.7) and (2.9) that

$$|P(z_r)| = M_r \quad \text{and} \quad |f(z) - P(z)| + |f'(z) - P'(z)| \leq r^{-n} M_r \quad \text{for} \quad z \in \Omega_r. \tag{2.10}$$

We can write $P(z) = P_1(z)P_2(z)$ where $P_1$ is the product of the terms $z - c_j$ over all zeros $c_j$ of $P$ with $|c_j| < 2r$, and is $1$ if there are no such $c_j$. Correspondingly, $P_2$ is a polynomial with all its zeros, if any, lying in $|z| \geq 2r$. Denoting by $C$ positive constants which are independent of $r$ this gives

$$M(\Omega_r, P_2'/P_2) \leq C/r, \quad M^* = M(\Omega_r, P_2) \leq C \min\{|P_2(z)| : z \in \Omega_r\}. \tag{2.11}$$

Also (2.10) yields

$$M_r \leq M(\Omega_r, P) \leq M^* \cdot M(\Omega_r, P_1) \leq M^*(3r)^d, \tag{2.12}$$

where $d \geq 0$ is the degree of $P_1$.

Let $z \in \Omega_r$ lie outside the union of the discs of centre $c_j$ and radius $r/\phi(r)$. Then (2.11) and (2.12) give

$$|P_1(z)| \geq \frac{r^d}{\phi(r)^d}, \quad |P(z)| = |P_1(z)P_2(z)| \geq \frac{M^* r^d}{C\phi(r)^d} \geq \frac{M_r}{C\phi(r)^d},$$

which on combination with (2.10) and (2.11) leads to

$$\begin{aligned}
\left|\frac{zf'(z)}{f(z)}\right| &= \left|\frac{zP'(z) + o(|P(z)|)}{P(z)(1+o(1))}\right| = \left|\frac{zP'(z)}{P(z)}(1+o(1)) + o(1)\right| \\
&\leq (1+o(1)\left|\frac{zP_1'(z)}{P_1(z)}\right| + (1+o(1))\left|\frac{zP_2'(z)}{P_2(z)}\right| + o(1) \leq ((n-1)+o(1))\phi(r).
\end{aligned}$$

$\square$

## 3 Deficiencies and the logarithmic derivative

We need the following lemma, which is a combination of [15, Lemma 4] (see also [14]) and Lemma 9 from [18].

**Lemma 3.1** *Let $f$ be transcendental and meromorphic of finite order in the plane, and set*

$$h(r) = r\frac{d}{dr}(T(r,f)) = \frac{1}{2\pi}\int_0^{2\pi} n(r,e^{i\phi},f)\,d\phi. \tag{3.1}$$

*For each large $r$ let $L_r$ be any measurable subset of $[0,2\pi)$ such that the Lebesgue measure of $L_r$ tends to $0$ as $r\to\infty$. Then there exists a subset $E_0$ of $[1,+\infty)$ of logarithmic density $1$ such that, as $r\to\infty$ in $E_0$,*

$$\int_{L_r}\left|\mathrm{Re}\left(\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right)\right|d\theta = o(h(r)). \tag{3.2}$$

*Suppose next that*

$$\delta(\infty,f) > 1-\sigma, \quad 0<\sigma<1, \quad K>1, \quad \sigma K<1. \tag{3.3}$$

*Then there exists a subset $E_1$ of $(1,+\infty)$, having lower logarithmic density $1-1/K$, such that for $r$ in $E_1$ we have*

$$(1-K\sigma)h(r) \leq I(r) = \frac{1}{2\pi}\int_0^{2\pi}\left|\mathrm{Re}\left(\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right)\right|d\theta. \tag{3.4}$$

Note that (3.1) is of course the classical Cartan formula [10, p.8] and that $h(r)$ tends to infinity since $f$ is transcendental. $\square$

## 4 Proof of Theorem 1.4

Let $f$ be a transcendental meromorphic function in the plane of order $\rho<1$, let $n\in\mathbb{N}$ and let $G_n$ be defined by (1.2), and assume that $G_n$ is transcendental. Lemma 2.4 gives a set $X_0\subseteq[1,\infty)$ of finite logarithmic measure such that (2.6) holds. Let the positive function $\phi(r)$ tend to infinity on $[1,\infty)$, and satisfy

$$\phi(r)=o(\log r), \quad \phi(r)=o(h(r)), \quad \phi(r)=o(n(r,f)+n(r,1/f)), \tag{4.1}$$

where $h(r)$ is defined by (3.1). This is certainly possible since $f$ is transcendental of order less than 1. For each large $r$, set

$$V_r = \left\{\theta\in[0,2\pi] : \left|\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}\right| > n\phi(r)\right\}. \tag{4.2}$$

Let $N$ be a large positive integer. For large $r>0$ let $r\beta(r)$ be the length of the longest arc of the circle $S(0,r)$ of centre $0$ and radius $r$ on which $|z^N G_n(z)|<1$, with $\beta(r)=2\pi$ if $|z^N G_n(z)|<1$ on all of $S(0,r)$. We begin with the following lemma.

**Lemma 4.1** *Suppose that $\beta(r) \to 2\pi$ on a set $Y_1$ of upper logarithmic density $\lambda \in (0,1)$. Then $\delta(\infty, f) \leq 1 - \lambda$.*

*Proof.* It may be assumed that the intersection of $Y_1$ with the exceptional set $X_0$ of (2.6) is empty, since this does not reduce the upper logarithmic density. Then by (2.6), (4.2), Lemma 2.5 and the fact that $N$ is large, the Lebesgue measure of $V_r$ satisfies $m(V_r) = o(1)$ on $Y_1$. Let

$$L_r = V_r \quad \text{if} \quad r \in Y_1, \quad L_r = \emptyset \quad \text{if} \quad r \notin Y_1.$$

It may be assumed further that $Y_1 \subseteq E_0$, where $E_0$ is as in Lemma 3.1, again since this does not reduce the upper logarithmic density. Thus Lemma 3.1 gives, using (4.1) and (4.2),

$$\int_0^{2\pi} \left| \mathrm{Re} \left( \frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} \right) \right| d\theta \leq O(\phi(r)) + o(h(r)) = o(h(r)) \tag{4.3}$$

for large $r \in Y_1$.

Now assume that $\delta(\infty, f) > 1 - \sigma > 1 - \lambda$. Then $\sigma > 0$ and we may choose $K > 1$ with $\sigma < 1/K < \lambda$. Hence (3.3) is satisfied and Lemma 3.1 implies that (3.4) holds on a set $E_1$ of lower logarithmic density at least $1 - 1/K > 1 - \lambda$, so that there must exist arbitrarily large $r \in Y_1 \cap E_1$. But for these $r$ the inequalities (3.4) and (4.3) give $(1 - K\sigma)h(r) \leq o(h(r))$, which is a contradiction. This proves Lemma 4.1. $\square$

We first prove part (i) of Theorem 1.4, and to this end we assume that $G_n$ has lower order $\mu < \alpha < 1/2$. This certainly holds if $\rho < 1/2$, because in this case $f$ may be written as a quotient of entire functions of order less than $1/2$ and a simple argument shows that the same is true of $G_n$. Assume further that $\delta(0, G_n) > 1 - \cos \pi\alpha$. Then by Lemma 2.2 there exists a subset $Y_1$ of $[1, \infty)$ having upper logarithmic density at least $1 - \mu/\alpha$ such that

$$\lim_{r \to \infty, r \in Y_1} r^N M(r, G_n) = 0,$$

which of course gives $\beta(r) = 2\pi$ for large $r$ in $Y_1$. Thus Lemma 4.1 implies at once that $\delta(\infty, f) \leq \mu/\alpha$, which completes the proof of part (i).

Parts (ii) and (iii) will now be proved together, so assume either that $\rho = 1/2$ and $\delta(\infty, f) = 1$, or that $f$ is entire of order $\rho$ with $\rho - 1/2$ small and positive, and in both cases that $G_n$ has finitely many zeros. Then there exists a rational function $R_0$ with at most a pole of order $N - 1$ at infinity such that

$$H(z) = \frac{1}{2z^N} \left( \frac{1}{G_n(z)} - R_0(z) \right) \tag{4.4}$$

is entire and transcendental, of order $\rho_1 \leq \rho$, and there exists $r_1 > 0$ such that

$$|z^N G_n(z)| < 1 \quad \text{for } |z| \geq r_1, \ |H(z)| > 1. \tag{4.5}$$

Let $\theta(r)$ be defined as in Lemma 2.3.

Suppose first that $\theta(r) \to 2\pi$ on a set $Y_1$ of upper logarithmic density $\lambda \in (0,1)$. This certainly holds under the hypotheses of part (ii), by the remarks following Lemma 2.3, and also applies for part (iii) if $H$ satisfies case (a) of Lemma 2.3. Then by (4.5) the hypotheses of Lemma 4.1 are satisfied, and so we have $\delta(\infty, f) \leq 1 - \lambda < 1$, which is a contradiction.

It therefore remains only to prove part (iii) in the case where the entire function $H$ satisfies conclusion (b) of Lemma 2.3. Let $M > 3$ and choose positive $\gamma$ and $\tau$ such that $\gamma$ is small and

$$\eta = \left( \frac{1-\gamma}{7M(\rho+1)} \right)^2 - 2\pi\tau > 0 \tag{4.6}$$

but $\eta$ is small. Since $f$ is entire in this case we may apply Lemma 2.1. This gives a subset $Q_M$ of $[1,\infty)$ satisfying (2.2) and (2.3), and there is no loss of generality in assuming that $Q_M \cap X_0 = \emptyset$, where $X_0$ is as in (2.6), as this assumption does not reduce the lower logarithmic density.

Let $F_\tau$ be as in Lemma 2.3. Then for large $r \in F_\tau \setminus X_0$ we have $\theta(r) > 2\pi(1-\tau)$ and $m(V_r) \leq 2\pi\tau + o(1)$, using (2.6), (4.2), (4.5), Lemma 2.5 and the fact that $N$ is large. By (2.1) and (4.1) we also have $m(U_r) \leq 2\pi\tau + o(1)$ for these $r$. Hence (2.3) and (4.6) show that the intersection $Q_M \cap F_\tau$ is bounded, which by (2.2) and (2.5) forces

$$1 - 2\rho_1(1-\tau) \leq \frac{3\tau}{M} \quad \text{and} \quad 2\rho - 1 \geq 2\rho_1 - 1 \geq \frac{\tau}{1-\tau}\left(1 - \frac{3}{M}\right) \geq \tau\left(1 - \frac{3}{M}\right).$$

Since $\rho < 1$ and $\gamma$ is small, while $\eta$ is small in (4.6), it follows that $\rho$ must satisfy

$$2\rho - 1 \geq \frac{1}{2\pi}\left(\frac{1}{14M}\right)^2 \left(1 - \frac{3}{M}\right) = q(M).$$

As noted in [4] the right hand side $q(M)$ in the last inequality has a maximum relative to the interval $(3,\infty)$ at $M = 9/2$, with $q(9/2) = 1/23814\pi$. This proves Theorem 1.4. □

# 5 Lemmas needed for Theorem 1.7

We need the following lemma from [20]. The result is closely related to [19, Lemma 2.4] and the method of proof is essentially the same.

**Lemma 5.1 ([20])** *Let $h$ be transcendental and meromorphic in the plane, of order less than $\rho < \infty$, and with finitely many poles. Let $(z_j)$ be a sequence in $\{z \in \mathbb{C} : |z| > 2\}$ such that $z_j \to \infty$ without repetition, and with exponent of convergence less than $\rho$. Let $M_1, M_2 \in \mathbb{R}$ be such that*

$$\rho + M_1 < 1, \quad M_2 \leq M_1 - 4\rho. \tag{5.1}$$

*For $m = 1, 2$, let $H_m$ be the union of the closures of the discs $B(z_j, |z_j|^{M_m})$.*

*Next, let $R_1$ be large and positive, such that*

$$h^{-1}(\{\infty\}) \subseteq B(0, R_1/2), \quad M(R_1, h) = \max\{|h(z)| : |z| = R_1\} > e^4, \tag{5.2}$$

*and*

$$\log|h(z)| \leq \left|\frac{z}{2}\right|^\rho \quad \text{for} \quad |z| \geq \frac{1}{2}R_1, \tag{5.3}$$

*as well as*

$$\left(\frac{1}{2}R_1\right)^{M_1 - M_2} > 4, \qquad \sum_{|z_j| > \frac{1}{2}R_1} 26|z_j|^{\rho + \frac{1}{2}(M_2 - M_1)} < 1. \tag{5.4}$$

*Let $w_0$ lie outside $H_1$, with*

$$|w_0| > R_1, \quad |h(w_0)| > T_1^2, \quad T_1 > M(R_1,h)^2, \tag{5.5}$$

*and let $C_0$ be the component of the set $\{z \in \mathbb{C} \setminus H_2 : |h(z)| > T_1\}$ in which $w_0$ lies. Then $C_0$ is unbounded.*

Note that (5.2), (5.3) and (5.4) hold for all sufficiently large $R_1$. Moreover, it follows from (5.1) and the fact that the sequence $(z_j)$ has exponent of convergence less than $\rho$ that the set of $r \geq 1$ for which the circle $S(0,r)$ meets $H_1$ has finite logarithmic measure. Hence there exist arbitrarily large $w_0 \notin H_1$ satisfying (5.5).

□

**Lemma 5.2** *Let $f$ be a transcendental meromorphic function in the plane, of order less than $1/6$. Assume that $G$ as defined by (1.3) has finitely many zeros. Then there exist a non-zero complex number $b$ and a set $E_0 \subseteq [1,\infty)$ of lower logarithmic density greater than $2/3$, such that $f(z) \sim b$ for all large $z$ with $|z| \in E_0$.*

*Proof.* Let $N$ be a large positive integer and choose $\rho$ with $\rho(f) < \rho < 1/6$. Since $G$ is transcendental [4, Lemma 2.1] and has finitely many zeros and order less than $\rho$, it follows from the $\cos \pi\rho$ theorem [12, Chapter 6] that there exists $E_0 \subseteq [1,\infty)$, with lower logarithmic density greater than $2/3$, such that

$$\lim_{r\to\infty, r\in E_0} r^N M(r,G) = 0. \tag{5.6}$$

Now define $h$ by

$$h(z) = \frac{1}{z^N G(z)}. \tag{5.7}$$

By Lemma 2 of [22] (see also [21, Lemma 4.1]) there exist arbitrarily large $T_1$ such that the length $L(r,T_1,h)$ of the level curves $|h(z)| = T_1$ lying in $B(0,r)$ satisfies

$$L(r,T_1,h) = O(r^2) \quad \text{as} \quad r \to \infty. \tag{5.8}$$

Here $T_1$ may be chosen so that, for additional convenience, the level curves $|h(z)| = T_1$ do not pass through the origin and have no multiple points. Hence these level curves may be parametrised locally in terms of $\arg h$, and for any given $w \in \mathbb{C}$ the stationary points of $\arg z$ and $\log|z-w|$ on these level curves form a discrete set. If this is not the case then either $|h(z)| \equiv T_1$ on a ray passing through the origin, which contradicts the choice of $T_1$, or $|h(z)| \equiv T_1$ on a circle of centre $w$, which is impossible since $h$ is transcendental.

Next, let $(z_j)$ be the set of all distinct zeros and poles of $f'$ with $r_j = |z_j| > 2$, and choose $\sigma$ and $M_1, M_2$ satisfying

$$\rho < \sigma < \frac{1}{6}, \quad M_1 = \sigma + \frac{2}{3}, \quad M_2 = \sigma. \tag{5.9}$$

This choice may be made so that no circle $S(z_j, r_j^\sigma)$ is tangent to a level curve $|h(z)| = T_1$. For $m = 1,2$, let $H_m$ be the union of the closures of the discs $B(z_j, |z_j|^{M_m})$. We assert that

$$\frac{f''(u)}{f'(u)} = o(1) \quad \text{for} \quad |u-z| \leq 1 \tag{5.10}$$

and for large $z \notin H_2$. To prove (5.10) let $R_0$ be large and positive and let $z \in \mathbb{C} \setminus H_2$ with $|z| > 2R_0 + 1$. Then the series expansion for $f''/f'$ gives, for $|u - z| \leq 1$,

$$\left| \frac{f''(u)}{f'(u)} \right| \leq \frac{n(R_0, f') + n(R_0, 1/f')}{|u| - R_0} + \sum_{|v_j| > R_0} \frac{1}{|u - v_j|},$$

where the $v_j$ are simply the $z_j$ but with repetition according to multiplicity. Since $z$ lies outside $H_2$ this yields

$$|u - v_j| \geq |z - v_j| - 1 \geq \frac{|v_j|^\sigma}{2}$$

for $|v_j| > R_0$ and so

$$\left| \frac{f''(u)}{f'(u)} \right| \leq \frac{n(R_0, f') + n(R_0, 1/f')}{R_0} + 2 \sum_{|v_j| > R_0} \frac{1}{|v_j|^\sigma} \to 0$$

as $R_0 \to \infty$, using (5.9). This proves (5.10), from which it follows that

$$f(z+1) - f(z) = \int_z^{z+1} f'(v)\, dv = \int_z^{z+1} f'(z)(1 + o(1))\, dv \sim f'(z)$$

and

$$\frac{f'(z)}{f(z)} \sim G(z) \tag{5.11}$$

for large $z \notin H_2$. Next, (5.9) implies that there exists a set $X_1$ of finite logarithmic measure with $S(0, r) \cap H_1 = \emptyset$ for $r \notin X_1$. In particular, (5.11) holds for large $z$ with $|z| \notin X_1$. It may be assumed that $X_1 \cap E_0 = \emptyset$, and it follows at once from (5.6) and (5.11) that

$$\lim_{r \to \infty, r \in E_0} \int_{S(0,r)} \left| \frac{f'(z)}{f(z)} \right| |dz| = 0. \tag{5.12}$$

The function $h$ is transcendental of order less than $\rho$ with finitely many poles, and the sequence $(z_j)$ has exponent of convergence less than $\rho$. Thus Lemma 5.1 may be applied to $h$, with $M_1, M_2$ given by (5.9) and hence satisfying (5.1). Let $R_1$ be large and positive, so large that (5.2), (5.3) and (5.4) hold, which is possible by (5.9). Choose $T_1$ and $w_0 \notin H_1$ as in (5.5), and such that (5.8) also holds. Let $C_0$ be the component determined in Lemma 5.1: then $C_0$ is unbounded.

Now choose a sequence $(s_m)$ such that

$$2s_m \leq s_{m+1} \leq s_m^3, \quad s_m \in E_0, \tag{5.13}$$

this being possible since $E_0$ has lower logarithmic density greater than $2/3$, and since $S(0, s_m)$ does not meet $H_2$ we may assume using (5.6) and (5.7) that $S(0, s_m) \subseteq C_0$ for all $m$. Now the part $Y_m$ of $\partial C_0$ lying in $s_m \leq |z| \leq s_{m+1}$ is contained in the union of the level set $|h(z)| = T_1$ and the circles $S(z_j, r_j^\sigma)$. The number of such circles which meet $Y_m$ is $O(s_{m+1})^\rho$ and their radii have sum $O(s_{m+1})^{\rho+\sigma} = o(s_{m+1})$. Hence the arc length of $Y_m$ is $O(s_{m+1})^2$ using (5.8).

We form a path $\gamma_m$ in the closure of $C_0$ joining $S(0, s_m)$ to $S(0, s_{m+1})$ as follows. First take a radial segment $\lambda$ given by $\arg z = \theta, s_m \leq |z| \leq s_{m+1}$, with $\theta$ chosen so that this segment is never tangent to any level curve $|h(z)| = T_1$, which may be done by the remark following (5.8), nor to any of the circles $S(z_j, r_j^\sigma)$. By construction $Y_m$ consists of a union of closed curves lying in $s_m < |z| < s_{m+1}$. Hence any arc of $\lambda$ which lies outside the closure of $C_0$ may be replaced by an arc of $Y_m$. Using (5.7), (5.11), (5.13) and the fact that $N$ is large we obtain

$$\int_{\gamma_m \cup S(0,s_m)} \left| \frac{f'(z)}{f(z)} \right| |dz| \leq \int_{\gamma_m \cup S(0,s_m)} \frac{2}{|z|^N T_1} |dz| = O(s_{m+1}^2 s_m^{-N}) = O(s_m^{-1}).$$

Hence from the union of the curves $\gamma_m$ and circles $S(0, s_m)$ a simple curve $\gamma$ may be constructed which tends to infinity and satisfies, by (5.13) again,

$$\int_\gamma \left| \frac{f'(z)}{f(z)} \right| |dz| < \infty.$$

Thus there exists a non-zero complex number $b$ such that $f(z) \to b$ as $z \to \infty$ on $\gamma$, which on combination with (5.12) gives the conclusion of the lemma. $\square$

It is perhaps worth remarking that the condition $\rho(f) < 1/6$ seems unlikely to be sharp in Theorem 1.7, but is required in our method in order to deduce Lemma 5.2 from Lemma 5.1. We need $M_2 > \rho(f)$ in order to achieve (5.10) for large $z$ outside $H_2$, so that in (5.1) the second inequality forces $5\rho(f) < M_1$ and the first inequality then requires $6\rho(f) < 1$.

# 6 Proof of Theorem 1.7

Let $f$ and $G$ be as in the statement of the theorem, and assume that $G$ has finitely many zeros. Then it follows from Lemma 5.2 that there exist a non-zero complex number $b$ and a set $E_0 \subseteq [1, \infty)$ of lower logarithmic density greater than $2/3$, such that $f(z) \sim b$ for all large $z$ with $|z| \in E_0$. Let

$$F = \frac{\Delta f}{f - b}.$$

Then $F$ must have infinitely many zeros, because otherwise Lemma 5.2 may also be applied to $f - b$ to give a non-zero constant $b_1$ and a set $E_1 \subseteq [1, \infty)$, again of lower logarithmic density greater than $2/3$, such that $f(z) - b \sim b_1$ for all large $z$ with $|z| \in E_1$, which is evidently impossible.

Let $z$ be large and a zero of $F$. Then $z$ is not a pole of $f$ because otherwise writing

$$G = F \cdot \frac{f - b}{f}$$

shows that $z$ is also a zero of $G$, contrary to the assumption that $G$ has finitely many zeros. But $\Delta f = F \cdot (f - b)$ and hence $z$ is a zero of $\Delta f$. This proves Theorem 1.7. $\square$

# References


[1] M. Ablowitz, R.G. Halburd and B. Herbst, On the extension of the Painlevé property to difference equations, Nonlinearity 13 (2000), 889-905.

[2] J.M. Anderson and J. Clunie, Slowly growing meromorphic functions, Comment. Math. Helv. 40 (1966), 267-280.

[3] P.D. Barry, On a theorem of Kjellberg, Quart. J. Math. Oxford (2) 15 (1964), 179-191.

[4] W. Bergweiler and J.K. Langley, Zeros of differences of meromorphic functions, Math. Proc. Camb. Phil. Soc. 142 (2007), 133-147.

[5] Y.M. Chiang and S.J. Feng, On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane, Ramanujan J. 16 (2008), 105-129.

[6] A. Eremenko, J.K. Langley and J. Rossi, On the zeros of meromorphic functions of the form $\sum_{k=1}^{\infty} \frac{a_k}{z-z_k}$, J. d'Analyse Math. 62 (1994), 271-286.

[7] A.A. Gol'dberg and O.P. Sokolovskaya, Some relations for meromorphic functions of order or lower order less than one, Izv. Vyssh. Uchebn. Zaved. Mat. 31 no.6 (1987), 26-31. Translation: Soviet Math. (Izv. VUZ) 31 no.6 (1987), 29-35.

[8] R.G. Halburd and R. Korhonen, Difference analogue of the lemma on the logarithmic derivative with applications to difference equations, J. Math. Anal. Appl. 314 (2006), 477-487.

[9] R.G. Halburd and R. Korhonen, Nevanlinna theory for the difference operator, Ann. Acad. Sci. Fenn. 31 (2006), 463-478.

[10] W.K. Hayman, Meromorphic functions, Oxford at the Clarendon Press, 1964.

[11] W.K. Hayman, The local growth of power series: a survey of the Wiman-Valiron method, Canad. Math. Bull. 17 (1974), 317-358.

[12] W.K. Hayman, Subharmonic functions Vol. 2, Academic Press, London, 1989.

[13] J. Heittokangas, R. Korhonen, I. Laine, J. Rieppo and K. Tohge, Complex difference equations of Malmquist type, Comput. Methods Funct. Theory 1 (2001), 27-39.

[14] S. Hellerstein, J. Miles and J. Rossi, On the growth of solutions of $f'' + gf' + hf = 0$, Trans. Amer. Math. Soc. 324 (1991), 693-706.

[15] S. Hellerstein, J. Miles and J. Rossi, On the growth of solutions of certain linear differential equations, Ann. Acad. Sci. Fenn. Ser. A. I. Math. 17 (1992), 343-365.

[16] J.D. Hinchliffe, The Bergweiler-Eremenko theorem for finite lower order, Result. Math. 43 (2003), 121-128.

[17] K. Ishizaki and N. Yanagihara, Wiman-Valiron method for difference equations, Nagoya Math. J. 175 (2004), 75-102.

[18] J.K. Langley, On differential polynomials, fixpoints and critical values of meromorphic functions, Result. Math. 35 (1999), 284-309.

[19] J.K. Langley, Composite Bank-Laine functions and a question of Rubel, Trans. Amer. Math. Soc. 354 (2002), 1177-1191.

[20] Jim Langley and Janis Meyer, A generalisation of the Bank-Laine property, Comput. Methods Funct. Theory 9 (2009), 213-225.

[21] J.K. Langley and John Rossi, Critical points of certain discrete potentials, Complex Variables 49 (2004), 621-637.

[22] J.K. Langley and D.F. Shea, On multiple points of meromorphic functions, J. London Math. Soc. (2) 57 (1998), 371-384.

[23] J. Miles and J. Rossi, Linear combinations of logarithmic derivatives of entire functions with applications to differential equations, Pacific J. Math. 174 (1996), 195-214.

[24] M. Tsuji, Potential theory in modern function theory, Maruzen, Tokyo, 1959.

[25] J.M. Whittaker, Interpolatory function theory, Cambridge Tract No. 33, Cambridge University Press 1935.

School of Mathematical Sciences, University of Nottingham, NG7 2RD, UK.
jkl@maths.nott.ac.uk