Deficient values of derivatives of meromorphic functions in the class S

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Abstract

It is shown that if f is transcendental and meromorphic in the plane with finitely many critical and asymptotic values then $\delta(b, f') = 0$ for every $b \in \mathbb{C} \setminus \{0\}$. If, in addition, f has finite lower order then the same result holds for higher derivatives.
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1 Introduction

Let f be a function transcendental and meromorphic in the plane. An important role in complex dynamics [1] is played by the singular values of the inverse function f^{-1}: these are the critical values of f and the asymptotic values, that is, values a such that $f(z)$ tends to a as z tends to infinity along a path γ_a. Denote by B the class of all functions f transcendental and meromorphic in the plane for which the set of finite singular values of f^{-1} is bounded, and by S the subclass consisting of those f for which f^{-1} has finitely many singular values. Both classes have been studied extensively in iteration theory [1, 6].

The class S is also of interest from the standpoint of value distribution theory. Thus Collingwood [5] showed that if f is a transcendental meromorphic function such that all finite singular values of f^{-1} lie in a finite set $\{a_j\}$ then any finite Nevanlinna deficient value of f must be one of the a_j. This result was generalized by Teichmüller [19], who proved for such f the asymptotic equality

$$m(r, 1/f') = \sum m(r, 1/(f - a_j)) + S(r, f)$$

in which, using standard notation from [9, 17], the term $S(r, f)$ denotes any quantity which is $o(T(r, f))$ outside a set of finite measure.

It was proved in [15] that if f is a transcendental meromorphic function in the class B and R is a rational function with a pole at infinity then

$$m(r, 1/(f-R)) = S(r, f).$$

In particular $\delta(0, f(z)-z) = 0$ and f has infinitely many fixpoints. In the present paper we prove some results on the deficient values of the derivatives of functions in the classes B and S. For functions of finite lower order in the class B we have the following.

Theorem 1.1 *Let f be transcendental and meromorphic of finite lower order in the plane such that the set of finite singular values of the inverse function f^{-1} is bounded. Let n be a positive integer. Then $\delta(b, f^{(n)}) = 0$ for every $b \in \mathbb{C} \setminus \{0\}$.*

Theorem 1.1 leads to a simple proof of a special case of a conjecture of Mues [16]: see §5. It seems difficult to extend the method of Theorem 1.1 to functions of infinite lower order, but we prove the following result for the class S.

Theorem 1.2 *Let f be transcendental and meromorphic in the plane such that the inverse function f^{-1} has finitely many singular values. Then $\delta(b, f') = 0$ for every $b \in \mathbb{C} \setminus \{0\}$.*

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2 Lemmas needed for Theorems 1.1 and 1.2

The following lemma, which was proved first in [6] for entire functions, has been used extensively in complex dynamics and value distribution theory: see also [1, 2, 15, 18].

Lemma 2.1 ([6, 18]) *Let f be transcendental and meromorphic in the plane such that the set of finite singular values of the inverse function f^{-1} is bounded. Then there exist $L > 0$ and $M > 0$ such that if $|z_0| > L$ and $|f(z_0)| > M$ then*

$$\left| z_0 \frac{f'(z_0)}{f(z_0)} \right| \geq \frac{\log |f(z_0)/M|}{C}, \tag{1}$$

in which C is a positive absolute constant, in particular independent of f, L and M.

The next lemma is a standard rescaled version of the distortion theorem for univalent functions in the unit disc [12, p.5, (1.3)].

Lemma 2.2 ([12]) *Let* $0 < r < R < \infty$ *and let* h *be analytic and univalent in the disc* $B(a, R)$*. Then*

$$\max\{|h'(z)| : |z-a| \leq r\} \leq \frac{2R^3}{(R-r)^3}|h'(a)| \leq \frac{16R^4}{(R-r)^4} \min\{|h'(z)| : |z-a| \leq r\}.$$

Proof. Assume without loss of generality that $a = 0$, and set $H(w) = h(Rw)/Rh'(0)$ and $z = Rw$. Then $H'(0) = 1$ and [12, p.5, (1.3)] gives, for $|z| \leq r$,

$$|H'(w)| \leq 2(1 - r/R)^{-3}, \quad |H'(w)| \geq (1 - r/R)/8$$

from which Lemma 2.2 follows at once.

We need Fuchs' small arcs lemma [7]: the version stated here is from [11, p.721].

Lemma 2.3 ([11]) *Let* g *be meromorphic in* $|z| \leq R$*, with* $g(0) = 1$*. Let* η_1, η_2 *be positive with* $\eta_1 + \eta_2 < 1$*. Then there exists a subset* E_R *of* $[0, R(1-\eta_1)]$*, having measure greater than* $R(1 - \eta_1 - \eta_2)$*, with the following property. If* $r \in E_R$ *and* F_r *is a subinterval of* $[0, 2\pi]$ *of length* m *then*

$$\int_{F_r} \left| \frac{rg'(re^{i\theta})}{g(re^{i\theta})} \right| d\theta \leq 400\eta_1^{-2}\eta_2^{-1}T(R, g)m \log \frac{2\pi e}{m}. \tag{2}$$

We require the following lemma, which for functions of finite lower order is completely standard.

Lemma 2.4 *Let* h *be transcendental and meromorphic in the plane, such that* $\delta(\infty, h) > 2\delta > 0$*. Then there exist a sequence* $r_k \to +\infty$ *and for each* k *an arc* Ω_k *of the circle* $S(0, r_k)$ *of centre* 0 *and radius* r_k*, such that*

$$\log |h(z)| > \delta T(r_k, h), \quad z \in \Omega_k, \tag{3}$$

and such that the angular measure m_k *of* Ω_k *satisfies*

$$m_k(\log T(r_k, h))^5 \to \infty. \tag{4}$$

If, in addition, h *has finite lower order* μ *then* $m_k \geq c$*, in which* c *is a positive constant depending only on* δ *and* μ*.*

Proof. The arc Ω_k is chosen so as to have $\log |h(z)| \geq 2\delta T(r_k, h)$ at one of its endpoints, and the angular measure m_k of Ω_k is determined by Lemma 2.3 as follows. Suppose first that h has finite lower order μ, and denote by c_j positive constants depending only on μ and δ. By [10] there exists a sequence $s_k \to +\infty$ with $T(2s_k, h) \leq c_1 T(s_k, h)$. Set $R = 2s_k$ and $\eta_1 = \eta_2 = 1/8$, and apply Lemma 2.3 to $h_1(z) = a_1 z^{n_1} h(z)$, with a_1, n_1 chosen so that $h_1(0) = 1$. Since

$$\left| \frac{h'(z)}{h(z)} \right| \leq \left| \frac{h_1'(z)}{h_1(z)} \right| + \frac{n_1}{|z|}, \quad T(2s_k, h_1) \leq T(2s_k, h) + O(\log s_k),$$

this gives $r_k \in E_R \cap [s_k, 2s_k]$ such that (2) becomes, for the interval F_{r_k} corresponding to the arc Ω_k,

$$\begin{aligned}
\int_{F_{r_k}} \left| \frac{r_k h'(r_k e^{i\theta})}{h(r_k e^{i\theta})} \right| d\theta &\leq c_2(T(2s_k, h) + O(\log s_k)) m_k \log \frac{2\pi e}{m_k} \\
&\leq c_3 T(r_k, h) m_k \log \frac{2\pi e}{m_k} < \delta T(r_k, h)
\end{aligned}$$

provided $m = m_k$ is chosen small enough and r_k is sufficiently large. But we have $\log |h(z)| \geq 2\delta T(r_k, h)$ at one endpoint of Ω_k, and (3) now follows.

The case of infinite lower order requires a result of Bergweiler and Bock [3, Lemma 1]. Let $\Phi(x) = \log T(e^x, h)$. Then by [3, Lemma 1] there exist sequences $x_k \to +\infty, M_k \to +\infty, \varepsilon_k \to 0+$ such that

$$\Phi(x_k + u) \leq \Phi(x_k) + \Phi'(x_k)u + \varepsilon_k, \quad |u| \leq \frac{M_k}{\Phi'(x_k)}.$$

Moreover [3, pp.328-9], the sequence (x_k) may be chosen so that $\Phi'(x_k)$ tends to infinity but satisfies $\Phi'(x_k) = o(\Phi(x_k))^{3/2}$. With $\rho_k = e^{x_k}$ this gives

$$T(r, h) \leq 2T(\rho_k, h) \left(\frac{r}{\rho_k} \right)^{\lambda_k}, \quad \left| \log \frac{r}{\rho_k} \right| \leq \frac{4}{\lambda_k}, \quad \lambda_k = \Phi'(x_k) = o(\log T(\rho_k, h))^{3/2}. \tag{5}$$

Let

$$R = R_k = \rho_k e^{4/\lambda_k}, \quad \eta_1 = 1 - e^{-2/\lambda_k} \sim \frac{2}{\lambda_k} = o(1), \quad \eta_2 = e^{-2/\lambda_k} - e^{-4/\lambda_k} \sim \frac{2}{\lambda_k}. \tag{6}$$

Then $R(1 - \eta_1 - \eta_2) = \rho_k$ and Lemma 2.3 and (5) and (6) then give r_k in $E_R \cap [\rho_k, R_k]$ such that (2) becomes

$$\begin{aligned}
\int_{F_{r_k}} \left| \frac{r_k h'(r_k e^{i\theta})}{h(r_k e^{i\theta})} \right| d\theta &\leq 400\lambda_k^3 (T(R_k, h) + O(\log R_k)) m \log \frac{2\pi e}{m} \\
&= o\left(T(r_k, h)(\log T(r_k, h))^{9/2} m \log \frac{2\pi e}{m} \right),
\end{aligned}$$

from which it is clear that $m = m_k$ may be chosen so as to satisfy (4).

3 Proof of Theorem 1.1

Suppose that f is as in the hypotheses but that $\delta(b, f^{(n)}) > 0$ for some positive integer n and some $b \in \mathbb{C} \setminus \{0\}$. Assume without loss of generality that $b = n!$.

By Lemma 2.4 there exist positive constants c_1, c_2 and a sequence $r_k \to +\infty$ such that

$$|f^{(n)}(z) - b| < \exp(-c_1 T(r_k, f^{(n)})), \quad z \in \Omega_k, \tag{7}$$

in which Ω_k is an arc of $S(0, r_k)$ of angular measure $m_k \geq c_2$. Integration of (7) gives a monic polynomial

$$P(z) = P_k(z) = z^n + \ldots = \prod_{j=1}^{n} (z - d_j), \tag{8}$$

possibly depending on r_k, such that

$$|f^{(q)}(z) - P^{(q)}(z)| < \exp(-c_3 T(r_k, f^{(n)})), \quad z \in \Omega_k, \quad q = 0, 1, \ldots, n, \tag{9}$$

denoting by $c_3, c_4, \ldots$ positive constants which do not depend on r_k.

With c_4 small, let r_k be large and choose

$$z_0 \in \Omega_k \setminus \bigcup_{|d_j| \leq 2r_k} B(d_j, c_4 r_k).$$

Then (9) gives

$$\log |f(z_0)| \geq n \log r_k - c_5. \tag{10}$$

Hence, using (8), (9) and (10),

$$\left| \frac{z_0 P'(z_0)}{P(z_0)} \right| = \left| \sum_{j=1}^{n} \frac{z_0}{z_0 - d_j} \right| = O(1), \quad \left| \frac{z_0 f'(z_0)}{f(z_0)} \right| = \left| \frac{z_0 P'(z_0) + o(1)}{P(z_0) + o(1)} \right| = O(1). \tag{11}$$

But (10) and (11) together contradict (1), and Theorem 1.1 is proved.

The example e^{e^z} shows that for infinite lower order the angular measure m_k cannot in general be expected to exceed $c_6 (\log T(r_k))^{-1}$, leading to at best $|z_0 f'(z_0)/f(z_0)| \leq c_7 \log T(r_k)$. Comparing this with (10) then suggests that the method of Theorem 1.1 cannot be extended to the case of infinite lower order.

4 Proof of Theorem 1.2

Let f be transcendental and meromorphic of infinite lower order in the plane, such that the inverse function f^{-1} has finitely many distinct finite singular values a_j. Let

$$\eta = \min\{|a_j - a_{j'}| : j \neq j'\}, \tag{12}$$

and let the positive constants L, M be as in Lemma 2.1. Assume without loss of generality that M is large.

Assume further that $\delta(1, f') > 0$, and apply Lemma 2.4. Then for large k there is an arc Ω_k of the circle $S(0, r_k)$, joining α_k to β_k, say, and of arc length

$$32\varepsilon_k = \frac{32}{(\log T(r_k, f'))^5}, \tag{13}$$

on which

$$|f'(z) - 1| < \exp(-c_1 T(r_k, f')). \tag{14}$$

Here and henceforth c_j, d_j, C_j denote positive constants not depending on r_k. From (13) and (14) it follows that

$$64\varepsilon_k \geq |f(\alpha_k) - f(\beta_k)| \geq 16\varepsilon_k \tag{15}$$

and since ε_k tends to 0 as $k \to \infty$ there is no loss of generality in assuming that

$$|f(\alpha_k) - a_j| \geq 8\varepsilon_k \quad \forall j. \tag{16}$$

Lemma 4.1 *Let ϕ be that branch of the inverse function f^{-1} mapping $f(\alpha_k)$ to α_k. Then ϕ extends to a univalent analytic function on $B(f(\alpha_k), 2\varepsilon_k)$ and satisfies there*

$$|\phi'(w) - 1| < \exp(-c_2 T(r_k, f')). \tag{17}$$

Proof. By (16) the function ϕ may be extended analytically and univalently to $B(f(\alpha_k), 8\varepsilon_k)$, and (14) gives

$$|\phi'(f(\alpha_k))| = \frac{1}{|f'(\alpha_k)|} \leq 2. \tag{18}$$

It follows from Lemma 2.2 that

$$|\phi'(w)| \leq c_3, \quad w \in B(f(\alpha_k), 4\varepsilon_k). \tag{19}$$

Furthermore, (15) shows that there exists a simple subarc L_k of $f(\Omega_k)$ joining $f(\alpha_k)$ to $S(f(\alpha_k), 4\varepsilon_k)$, and (14) gives

$$|\phi'(w) - 1| < 2\exp(-c_1 T(r_k, f')) \tag{20}$$

on L_k. Using (19), (20) and the standard harmonic measure estimate

$$\omega(w, L_k, B(f(\alpha_k), 4\varepsilon_k) \setminus L_k) \geq c_4, \quad w \in B(f(\alpha_k), 2\varepsilon_k) \setminus L_k,$$

the estimate (17) follows by applying the two constants theorem [17, p.42] to the subharmonic function $\log |\phi'(w) - 1|$. This proves Lemma 4.1.

Lemma 4.2 *Let σ be positive but small compared to $\min\{1, \eta\}$. Then provided r_k is large enough there exists ζ_k with*

$$|\zeta_k - f(\alpha_k)| = \sigma \tag{21}$$

such that ϕ can be continued analytically and univalently to $B(\zeta_k, \sigma)$, and satisfies there

$$|\phi'(w) - 1| < \exp\left(\frac{-d_1 T(r_k, f')}{(\log T(r_k, f'))^{10}}\right) \tag{22}$$

and

$$|\phi(w) - \alpha_k| < 1. \tag{23}$$

Proof. Let a_ν be the nearest a_j to $f(\alpha_k)$. Choose ζ_k satisfying (21) so that $a_\nu, f(\alpha_k), \zeta_k$ are collinear, with $f(\alpha_k)$ separating a_ν from ζ_k.

We assert that no a_j lies in $B(\zeta_k, \sigma + 2\varepsilon_k)$. Assuming the existence of such an a_μ gives $|f(\alpha_k) - a_\mu| < 3\sigma$ and so $a_\mu = a_\nu$ by (12), since σ is small compared to η. This is a contradiction since $|a_\nu - \zeta_k| \geq \sigma + 8\varepsilon_k$ by (16) and the choice of ζ_k.

Hence ϕ can be continued analytically and univalently into $B(\zeta_k, \sigma+2\varepsilon_k)$, starting at $f(\alpha_k)$. By (18), (21) and Lemma 2.2, ϕ satisfies

$$|\phi'(w)| \leq \frac{d_2}{\varepsilon_k^4}, \quad w \in B(\zeta_k, \sigma + \varepsilon_k). \tag{24}$$

Moreover the disc $B(f(\alpha_k), 2\varepsilon_k)$ intersects the circle $S(\zeta_k, \sigma + \varepsilon_k)$ in an open arc Σ_k of angular measure at least $d_3\varepsilon_k$, and (17) gives

$$|\phi'(w) - 1| < \exp(-c_2 T(r_k, f')), \quad w \in \Sigma_k. \tag{25}$$

Poisson's formula yields

$$\omega(w, \Sigma_k, B(\zeta_k, \sigma + \varepsilon_k)) \geq d_4\varepsilon_k^2$$

for $w \in B(\zeta_k, \sigma)$, and so (22) follows from (13), (24), (25), on applying the two constants theorem again to $\log|\phi'(w) - 1|$. Since $\phi(f(\alpha_k)) = \alpha_k$, integration of (22) starting from $f(\alpha_k) \in \partial B(\zeta_k, \sigma)$ gives (23). This proves Lemma 4.2.

Lemma 4.3 *Let τ be positive, but small compared to σ/q, where q is the number of finite singular values a_j. Choose*

$$y_k \in [\mathrm{Im}(\zeta_k) - \sigma/4, \mathrm{Im}(\zeta_k) + \sigma/4] \tag{26}$$

such that the strip $\{w \in \mathbb{C} : |\mathrm{Im}(w) - y_k| < 4\tau\}$ contains none of the a_j. Then provided r_k is large enough ϕ may be continued analytically and univalently to the domain

$$D_k = \{w \in \mathbb{C} :\ |\mathrm{Re}(w) - \mathrm{Re}(\zeta_k)| < 8M,\ |\mathrm{Im}(w) - y_k| < \tau\}, \tag{27}$$

starting from the point

$$W_k = \mathrm{Re}(\zeta_k) + iy_k \in B(\zeta_k, \sigma), \tag{28}$$

and satisfies, for $w \in D_k$,

$$|\phi'(w) - 1| < \exp\left(\frac{-C_1 T(r_k, f')}{(\log T(r_k, f'))^{10}}\right) \tag{29}$$

and

$$|\phi(w) - \alpha_k| < 1 + 16M. \tag{30}$$

Proof. By the choice of y_k and τ the function ϕ may in fact be continued analytically and univalently to

$$D_k'' = \{w \in \mathbb{C} : |\mathrm{Re}(w) - \mathrm{Re}(\zeta_k)| < 32M, \ |\mathrm{Im}(w) - y_k| < 4\tau\},$$

which has centre W_k. Since (22) gives $|\phi'(W_k)| \leq 2$, repeated application of Lemma 2.2 on discs of radius C_2 shows that

$$|\phi'(w)| \leq C_3, \quad w \in D_k' = \{w \in \mathbb{C} : |\mathrm{Re}(w) - \mathrm{Re}(\zeta_k)| < 16M, \ |\mathrm{Im}(w) - y_k| < 2\tau\}.$$

Since (22) holds on the vertical arc $w = \mathrm{Re}(\zeta_k) + iy$, $|y - y_k| \leq 2\tau$, the estimate (29) follows from the two constants theorem, and again (30) is obtained by integration, starting from W_k and using (23). This proves Lemma 4.3.

Let

$$A_k = W_k - 4M - \frac{r_k}{8}, \quad B_k = W_k + 4M + \frac{r_k}{8}, \quad U_k = B\left(A_k, \frac{r_k}{8}\right), \quad V_k = B\left(B_k, \frac{r_k}{8}\right).$$

Then since $\mathrm{dist}\{U_k, V_k\} = 8M$ at least one of the discs U_k and V_k lies in the region $\{w \in \mathbb{C} : |w| > 4M\}$. Assume that this is true of U_k, the subsequent argument in the contrary case being identical.

Lemma 4.4 *The function ϕ may be continued analytically and univalently to U_k, with*

$$\phi'(w) = 1 + o(1), \quad |\phi(w) - \alpha_k| \leq 2 + 16M + \frac{r_k}{4}, \quad w \in U_k. \tag{31}$$

Proof. Since U_k lies in $\{w \in \mathbb{C} : |w| > 4M\}$ the function ϕ extends analytically and univalently to $B(A_k, r_k/8 + 2M)$, starting at $W_k - 4M \in D_k$. Then we have $|\phi'(W_k - 4M)| \leq 2$ by (29), and Lemma 2.2 gives

$$|\phi'(w)| \leq C_4 r_k^4, \quad w \in B(A_k, r_k/8 + M). \tag{32}$$

Also D_k intersects $S(A_k, r_k/8 + M)$ in an arc Γ_k of angular measure at least $C_5 r_k^{-1}$, so that

$$\omega(w, \Gamma_k, B(A_k, r_k/8 + M)) \geq \frac{C_6}{r_k^2}, \quad w \in U_k. \tag{33}$$

But (29) holds on Γ_k, and so the two constants theorem and (32) and (33) give

$$|\phi'(w) - 1| \leq \exp\left(\frac{-C_7 T(r_k, f')}{r_k^2 (\log T(r_k, f'))^{10}}\right) = \frac{o(1)}{r_k}, \quad w \in U_k, \tag{34}$$

since f has infinite lower order and consequently so has f'. Integrating (34) starting from the point $W_k - 4M \in D_k \cap \partial U_k$, together with (30), leads to the second estimate of (31). This proves Lemma 4.4.

To complete the proof of Theorem 1.2 choose $w_0 \in U_k$ with $|w_0| \geq r_k/16$. Since $|\alpha_k| = r_k$, the estimate (31) gives, provided r_k is large enough,

$$L < r_k/2 \leq |z_0| \leq 3r_k/2, \quad z_0 = \phi(w_0).$$

But now (1) and (31) give

$$\frac{1}{48} = \frac{(r_k/16)(1/2)}{3r_k/2} \leq \left| w_0 \frac{\phi'(w_0)}{\phi(w_0)} \right| = \left| \frac{f(z_0)}{z_0 f'(z_0)} \right| \leq \frac{C}{\log|w_0/M|} \leq \frac{C}{\log(r_k/16M)},$$

which is obviously impossible if r_k is large enough.

We conclude this section by noting that the method of Theorem 1.2 appears to break down for higher derivatives. If $\delta(2, f'') > 0$ then $\phi = f^{-1}$ can in principle be continued to some region U, with $f(z)$ close to a monic quadratic on $\phi(U)$. However, it seems difficult to exclude the possibility that ϕ' is small on U, so that $\phi(U)$ has small diameter, which may in turn preclude obtaining a good upper bound for $|zf'(z)/f(z)|$.

5 A special case of the Mues conjecture

Let f be transcendental and meromorphic in the plane and let n be a positive integer. It was observed by Hayman [8, 9] that since all poles of $f^{(n)}$ have multiplicity at least $n+1$ we have

$$\sum_{a \in \mathbb{C}} \delta(a, f^{(n)}) \leq \frac{n+2}{n+1}.$$

This was improved to

$$\sum_{a \in \mathbb{C}} \delta(a, f^{(n)}) \leq \frac{n^2+5n+4}{n^2+4n+2}$$

by Mues [16]. If, in addition, all poles of f are simple then Mues [16] showed that

$$\sum_{a \in \mathbb{C}} \delta(a, f^{(n)}) \leq 1,$$

and conjectured that this remains true in the general case. The strongest general result appears to be [14, 23] (see also [20, 21, 22])

$$\sum_{a \in \mathbb{C}} \delta(a, f^{(n)}) \leq \frac{2n+2}{2n+1}.$$

We prove here that if f has finite lower order and $f^{(n)}$ has a finite Picard value a, that is, a finite value taken finitely often in the plane, then $f^{(n)}$ has no other finite deficient value.

Theorem 5.1 *Let f be transcendental and meromorphic of finite lower order in the plane, let n be a positive integer, and assume that $f^{(n)}$ has a finite Picard value a. Then $\delta(b, f^{(n)}) = 0$ for all $b \in \mathbb{C} \setminus \{a\}$.*

Proof. Set $g(z) = f^{(n-1)}(z) - az$. Then g is transcendental and meromorphic of finite lower order, and $g' = f^{(n)} - a$ has finitely many zeros. By Hinchliffe's extension [13] to functions of finite lower order of a result of Bergweiler and Eremenko [4], the function g has finitely many asymptotic values and so is in the class S. The result then follows from Theorem 1.1.

References

[1] W. Bergweiler, Iteration of meromorphic functions, Bull. Amer. Math. Soc. 29 (1993), 151-188.

[2] W. Bergweiler, On the zeros of certain homogeneous differential polynomials, Arch. Math. (Basel) 64 (1995), 199-202.

[3] W. Bergweiler and H. Bock, On the growth of meromorphic functions of infinite order, J. d'Analyse Math. 64 (1994), 327-336.

[4] W. Bergweiler and A. Eremenko, On the singularities of the inverse to a meromorphic function of finite order, Rev. Mat. Iberoamericana 11 (1995), 355-373.

[5] E.F. Collingwood, Sur les valeurs exceptionelles des fonctions entières d'ordre fini, C.R. Acad. Sci. Paris 179 (1924), 1125-1127.

[6] A. Eremenko and M.Yu. Lyubich, Dynamical properties of some classes of entire functions, Ann. Inst. Fourier Grenoble 42 (1992), 989-1020.

[7] W.H.J. Fuchs, Proof of a conjecture of G. Pólya concerning gap series, Illinois J. Math. 7 (1963), 661-667.

[8] W.K. Hayman, Picard values of meromorphic functions and their derivatives, Ann. of Math. 70 (1959), 9-42.

[9] W.K. Hayman, Meromorphic functions, Oxford at the Clarendon Press, 1964.

[10] W.K. Hayman, On the characteristic of functions meromorphic in the plane and of their integrals, Proc. London Math. Soc. (3) 14A (1965), 93-128.

[11] W.K. Hayman, Subharmonic functions Vol. 2, Academic Press, London, 1989.

[12] W.K. Hayman, Multivalent functions, 2nd edition, Cambridge Tracts in Mathematics 110, Cambridge University Press, Cambridge 1994.

[13] J.D. Hinchliffe, The Bergweiler-Eremenko theorem for finite lower order, Result. Math. 43 (2003), 121-128.

[14] K. Ishizaki, Some remarks on results of Mues about deficiency sums of derivatives, Arch. Math. (Basel) 55 (1990), 374-379.

[15] J.K. Langley and J.H. Zheng, On the fixpoints, multipliers and value distribution of certain classes of meromorphic functions, Ann. Acad. Sci. Fenn. 23 (1998), 133-150.

[16] E. Mues, Über eine Defekt- und Verzweigungsrelation für die Ableitung meromorpher Funktionen, Manuscripta Math. 5 (1971), 275-297.

[17] R. Nevanlinna, Eindeutige analytische Funktionen, 2. Auflage, Springer, Berlin, 1953.

[18] P.J. Rippon and G.M. Stallard, Iteration of a class of hyperbolic meromorphic functions, Proc. Amer. Math. Soc. 127 (1999), 3251-3258.

[19] O. Teichmüller, Eine Umkehrung des zweiten Hauptsatzes der Wertverteilungslehre, Deutsche Mathematik 2 (1937), 96-107.

[20] Y.F. Wang, Total defect of a meromorphic function and all its derivatives, Result. Math. 22 (1992), 631-638.

[21] Y.F. Wang, Total defect of a meromorphic function and all its derivatives II, Result. Math. 24 (1993), 190-199.

[22] Y.F. Wang, On Mues' conjecture and Picard values, Sci. China Ser. A 36 (1993), 28-35.

[23] L. Yang, Precise estimate of total deficiency of meromorphic derivatives, J. d'Analyse Math. 55 (1990), 287-296.

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