NONVANISHING DERIVATIVES AND NORMAL FAMILIES

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ABSTRACT. We consider the differential operators Ψ_k defined by $\Psi_1(y) = y$ and $\Psi_{k+1}(y) = y\Psi_k(y) + \frac{d}{dz}(\Psi_k(y))$ for $k \in \mathbb{N}$. We show that if F is meromorphic in $\mathbb{C}$ and $\Psi_k(F)$ has no zeros for some $k \geq 3$, and if the residues at the simple poles of F are not positive integers, then F has the form $F(z) = ((k-1)z+\alpha)/(z^2+\beta z+\gamma)$ or $F(z) = 1/(\alpha z + \beta)$ where $\alpha, \beta, \gamma \in \mathbb{C}$. If the residues at the simple poles of F are bounded away from zero, then this also holds for $k = 2$. We further show that under suitable additional conditions a family of meromorphic functions F is normal if each $\Psi_k(F)$ has no zeros. These conditions are satisfied, in particular, if there exists $\delta > 0$ such that $\mathrm{Re}(\mathrm{Res}(F, a)) \leq -\delta$ for all poles a of each F in the family. Using the fact that $\Psi_k(f'/f) = f^{(k)}/f$ we deduce in particular that if f and $f^{(k)}$ have no zeros for all f in some family $\mathcal{F}$ of meromorphic functions, where $k \geq 2$, then $\{f'/f : f \in \mathcal{F}\}$ is normal.

1. INTRODUCTION AND RESULTS

The following result was conjectured by Hayman [8, p. 23] in 1959 and proved by Frank [4] in 1976 for $k \geq 3$ and by the second author [12] in 1993 for $k = 2$.

Theorem A. *Let f be meromorphic in $\mathbb{C}$ and let $k \geq 2$. Suppose that f and $f^{(k)}$ have no zeros. Then f has the form $f(z) = e^{az+b}$ or $f(z) = (az+b)^{-n}$, where $a, b \in \mathbb{C}$, $a \neq 0$, and $n \in \mathbb{N}$.*

In the case where f is entire the result was proved by Hayman [8, Theorem 5] himself for $k = 2$ and by Clunie [2] in the general case: see also [9, p. 67]. In this case f'/f is constant under the hypotheses of Theorem A.

A heuristic principle attributed to Bloch says that the family of all functions meromorphic and possessing a given property in some domain is likely to be normal, if every function meromorphic in the plane with the same property is constant; see [16, 20] for a thorough discussion of Bloch's principle. The following result of Schwick [17, Theorem 5.1] can be considered as the normal families analogue arising according to Bloch's principle from Theorem A restricted to entire functions.

Theorem B. *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions holomorphic in a domain Ω. Suppose that f and $f^{(k)}$ have no zeros in Ω, for all $f \in \mathcal{F}$. Then $\{f'/f : f \in \mathcal{F}\}$ is normal.*

It was shown by the first author [1, Theorem 3] that the conclusion of Theorem B remains valid for families of meromorphic functions if $k = 2$.

One of the questions that motivated this paper was whether Schwick's Theorem B can also be extended to families of meromorphic functions if $k \geq 3$. It turns out that

Date: January 10, 2003.

1991 *Mathematics Subject Classification.* 30D35, 30D45.

The first author is supported by the German-Israeli Foundation for Scientific Research and Development G.I.F., G -643-117.6/1999, and INTAS-99-00089. The second author thanks the DAAD for supporting a visit to Kiel in June-July 2002. Both authors thank Günter Frank for helpful discussions.

this is in fact the case: see Corollary 1.1. The method used has led to considerable generalizations of Theorems A and B.

In order to state these generalizations, we define differential operators Ψ_k for $k \in \mathbb{N}$ by

$$\Psi_1(y) = y, \quad \Psi_{k+1}(y) = y\Psi_k(y) + \frac{d}{dz}(\Psi_k(y)). \tag{1.1}$$

The connection to Theorems A and B is given by the following lemma, which is easily proved by induction.

Lemma 1.1. *Let f be meromorphic on a domain Ω and let $F = f'/f$. Then for each $k \in \mathbb{N}$ we have $\Psi_k(F) = f^{(k)}/f$.*

Thus Theorem A is equivalent to the statement that if F is a function of the form $F = f'/f$, where f is meromorphic in $\mathbb{C}$ and has no zeros, and if $\Psi_k(F)$ has no zeros, then F is constant or of the form $F(z) = -n/(z+c)$ with $n \in \mathbb{N}$ and $c \in \mathbb{C}$. We note that a meromorphic function F is of the form $F = f'/f$ for some meromorphic function f with no zeros if and only if all poles of F are simple, with negative integers as residues.

Theorem 1.1. *Let $k \geq 3$ be an integer, and let F be meromorphic and non-constant in the plane and satisfy both of the following conditions:*

(i) *$\Psi_k(F)$ has no zeros;*
(ii) *if a is a simple pole of F then* $\text{Res}(F, a) \notin \{1, \ldots, k-1\}$.

Then F has the form

$$F(z) = \frac{(k-1)z + \alpha}{z^2 + \beta z + \gamma} \tag{1.2}$$

or

$$F(z) = \frac{1}{\alpha z + \beta}. \tag{1.3}$$

Here $\alpha, \beta, \gamma \in \mathbb{C}$, with $\alpha \neq 0$ in (1.3).

Conversely, if F has the form (1.2) *or* (1.3), *and if* (*ii*) *holds, then $\Psi_k(F)$ has no zeros. If F has the form* (1.2) *or* (1.3), *but* (*ii*) *does not hold, then $\Psi_k(F) \equiv 0$.*

The conclusion of Theorem 1.1 does not hold for $k = 2$, as shown by the example $F = 1/g$ where g is a transcendental entire function such that $g' - 1$ has no zeros. Then

$$\Psi_2(F) = F' + F^2 = \frac{1 - g'}{g^2} \neq 0.$$

Thus F satisfies (i) and (ii).

The conclusion of Theorem 1.1 can be obtained in the case $k = 2$, however, with an additional hypothesis.

Theorem 1.2. *Let F be meromorphic and non-constant in the plane, such that:*

(i) *$\Psi_2(F) = F' + F^2$ has no zeros;*
(ii) *if a is a simple pole of F then* $\text{Res}(F, a) \neq 1$;
(iii) *there exists $\delta > 0$ such that if a is a simple pole of F then* $|\text{Res}(F, a)| \geq \delta$.

Then F has the form (1.2) *with $k = 2$ or the form* (1.3).

Again we find that if F has the form (1.2) with $k = 2$ or the form (1.3), then $\Psi_k(F)$ has no zeros if $\operatorname{Res}(F, a) \neq 1$ for each simple pole a of F, while $\Psi_k(F) \equiv 0$ otherwise.

We turn next to normal family analogues of Theorems 1.1 and 1.2, thereby generalizing Theorem B: that is, we consider to what extent the condition $\Psi_k(f) \neq 0$ for all functions f in some family implies normality. First we note that the family of all functions F of the form (1.2) or (1.3) is not normal. On the other hand, the family of all functions F of the form (1.3) satisfying condition (iii) with the same δ is normal. In order to introduce a condition to deal with functions of the form (1.2) we observe that if F has this form, then

$$\sum_{a \in F^{-1}(\{\infty\})} \operatorname{Res}(F, a) = k - 1 \tag{1.4}$$

by the residue theorem.

We use the notation $D(c, R) = \{z \in \mathbb{C} : |z - c| < R\}$ for $c \in \mathbb{C}$ and $R > 0$.

Theorem 1.3. *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions meromorphic in a domain Ω. Suppose that there exists $\delta \in (0, 1]$ such that the following conditions hold for all $F \in \mathcal{F}$:*

(i) *$\Psi_k(F)$ has no zeros;*
(ii) *if a is a simple pole of F then $|\operatorname{Res}(F, a) - j| \geq \delta$ for $j \in \{0, 1, \ldots, k-1\}$;*
(iii) *if $c \in \Omega$ and $R > 0$ with $D(c, R) \subset \Omega$, if $D(c, \delta R)$ contains two poles of F, counting multiplicities, and if $D(c, R) \setminus D(c, \delta R)$ contains no poles of F, then*

$$\left| \sum_{a \in D(c, \delta R)} \operatorname{Res}(F, a) - (k - 1) \right| \geq \delta. \tag{1.5}$$

Then $\mathcal{F}$ is normal.

If F has two distinct poles $a, b \in D(c, \delta R)$ in (iii), then (1.5) takes the form $|\operatorname{Res}(F, a) + \operatorname{Res}(F, b) - (k - 1)| \geq \delta$. If F has a double pole $a \in D(c, \delta R)$ in (iii), then (1.5) takes the form $|\operatorname{Res}(F, a) - (k - 1)| \geq \delta$. This means that the inequality in (ii) is also required for double poles a if $j = k - 1$.

We note that conditions (ii) and (iii) in Theorem 1.3 are satisfied if we have $\operatorname{Re}(\operatorname{Res}(F, a)) \leq -\delta$ for all poles a of F. In particular, this is the case if $F = f'/f$ for some meromorphic function f without zeros.

Combining this observation with Lemma 1.1 we obtain the following corollary to Theorem 1.3, which extends Theorem B to families of meromorphic functions.

Corollary 1.1. *Let $k \geq 2$ and let $\mathcal{F}$ be a family of functions meromorphic in a domain Ω. Suppose that f and $f^{(k)}$ have no zeros in Ω, for all $f \in \mathcal{F}$. Then $\{f'/f : f \in \mathcal{F}\}$ is normal.*

We will prove Theorems 1.1–1.3 in §§2–4 and make some additional remarks in §5.

2. Proof of Theorem 1.1

Our proof is based on a method of Frank [4, 5, 6, 7]. We start with the following lemma.

Lemma 2.1. *Let $k \geq 2$ be an integer. Let y be meromorphic on a domain Ω, such that if a is a simple pole of y then* $\mathrm{Res}(y, a) \notin \{1, \dots, k-1\}$. *Let $n \in \mathbb{N}$ with $n \leq k$. If y has a pole at a of multiplicity m then $\Psi_n(y)$ has a pole at a of multiplicity nm.*

Proof. The lemma is trivially true for $n = 1$. Suppose first that $m \geq 2$, that $k > n \geq 1$, and that y and $\Psi_n(y)$ have poles at a of multiplicity m and nm respectively. Then $y\Psi_n(y)$ has a pole of multiplicity $(n+1)m$, while $(\Psi_n(y))'$ has a pole of multiplicity $nm + 1 < (n+1)m$. Using (1.1), $\Psi_{n+1}(y)$ has a pole of multiplicity $(n+1)m$ as required.

Suppose next that a is a simple pole of y with residue b. We assert that

$$\Psi_n(y) = \frac{b(b-1)\dots(b-n+1)}{(z-a)^n} + O(|z-a|^{1-n}), \quad z \to a, \tag{2.1}$$

for $n = 1, \dots, k$. This is obviously true for $n = 1$, and we assume that (2.1) holds for some n with $1 \leq n < k$. Then as $z \to a$ we obtain, using (1.1),

$$\Psi_{n+1}(y) = \left(\frac{b(b-1)\dots(b-n+1)}{(z-a)^n}\right)\left(\frac{b}{z-a} - \frac{n}{z-a}\right) + O(|z-a|^{-n}),$$

which gives (2.1) with n replaced by $n+1$. Since $b \notin \{0, 1, \dots, k-1\}$, (2.1) shows that each $\Psi_n(y)$, for $1 \leq n \leq k$, has a pole at a of multiplicity n. □

Assume now that $k \geq 3$ and that F is meromorphic and non-constant in the plane, such that (i) and (ii) hold. Define $M = \Psi_k(F)$.

Lemma 2.2. *There exist entire functions g, h with*

$$M = g^{-k}, \quad h = -Fg. \tag{2.2}$$

Proof. The existence of an entire g as in (2.2) follows at once from (i) of Theorem 1.1 and Lemma 2.1. Moreover, g has a zero of multiplicity m whenever F has a pole of multiplicity m, and so h is also entire. □

Frank's method requires auxiliary functions as defined in the next lemma: the notation used here is in accordance with [5, 7].

Lemma 2.3. *Define functions f_j, w_j for $j = 1, \dots, k$ by*

$$f_j(z) = z^{j-1}, \quad w_j(z) = f_j'(z)g(z) + f_j(z)h(z). \tag{2.3}$$

Then the w_j are entire functions and form a fundamental solution set of a linear differential equation

$$w^{(k)} + \sum_{q=0}^{k-2} A_q w^{(q)} = 0, \tag{2.4}$$

in which the coefficients A_q are entire functions with

$$T(r, A_q) = O(\log r + \max\{\log^+ T(r, w_j)\}) = O(\log rT(r, F)) \quad (n.e.). \tag{2.5}$$

Proof. We follow Frank's Wronskian method. In a simply connected domain Ω avoiding poles of F we define f by $f'/f = F$. Then Lemmas 1.1 and 2.2 give $M = f^{(k)}/f$ and

$$W(f_1, \dots, f_k, f) = W(f_1, \dots, f_k)f^{(k)} = c_k f^{(k)} = c_k M f = c_k f g^{-k}, \tag{2.6}$$

with c_k a non-zero constant. Standard properties of Wronskians [11, Chapter 1] give

$$c_k(fg)^{-k} = W(f_1/f, \dots, f_k/f, 1) = (-1)^k W((f_1/f)', \dots (f_k/f)') \tag{2.7}$$

and, because $w_j = fg(f_j/f)'$,

$$W(w_1, \dots, w_k) = (-1)^k c_k. \tag{2.8}$$

Thus the w_j, which are plainly entire, are linearly independent solutions of an equation (2.4), and (2.5) is a standard estimate [11, Lemma 7.7]. □

The following is a special case of a lemma which is fundamental to Frank's method, and which in its present form may be found in [5, Lemma 6].

Lemma 2.4. *Let $k \in \mathbb{Z}, k \geq 3$ and let f_j be as in (2.3). Let $G, H, A_0, \dots, A_{k-2}$ be meromorphic on a domain Ω. Then the functions $f_1 H + f_1' G, \dots, f_k H + f_k' G$ are solutions in Ω of the equation (2.4) if and only if, setting $A_k = 1$ and $A_{k-1} = A_{-1} = a_{-1} = 0$ and, for $0 \leq \mu \leq k$,*

$$M_{k,\mu}(w) = \sum_{m=\mu}^{k} \frac{m!}{\mu!(m-\mu)!} A_m w^{(m-\mu)}, \quad M_{k,-1}(w) = 0,$$

we have, for $0 \leq \mu \leq k-1$,

$$M_{k,\mu}(H) = -M_{k,\mu-1}(G). \tag{2.9}$$

Using Lemma 2.4, we prove next:

Lemma 2.5. *F is a rational function.*

Proof. We follow Frank's method, in the form used in [5] and, in particular, in [7]. Apply Lemma 2.4 to the w_j. It follows that g and h solve a system of equations

$$T_\mu(g) = S_\mu(h) = \sum_{j=0}^{k-\mu} c_{j,\mu} h^{(j)}, \quad 0 \leq \mu \leq k-1, \tag{2.10}$$

in which T_μ and S_μ are linear differential operators with coefficients λ_ν which are rational functions in the A_j and their derivatives and by (2.5) satisfy

$$T(r, \lambda_\nu) = O(\log rT(r, F)) \quad (n.e.). \tag{2.11}$$

In particular, $\mu = k-1$ gives

$$h' = U(g) = -(k-1)g''/2 - A_{k-2}g/k. \tag{2.12}$$

We distinguish two cases.

Case 1. Here we assume that the coefficient of h in at least one S_μ in (2.10) is not identically zero.

Let ν be the largest integer with $0 \leq \nu \leq k-1$ such that $c_{0,\nu} \not\equiv 0$. Then (2.2), (2.10) and (2.12) give

$$h = -Fg = (c_{0,\nu})^{-1} \left(T_\nu(g) - \sum_{j=1}^{k-\nu} c_{j,\nu} \frac{d^{j-1}}{dz^{j-1}}(U(g)) \right) = V(g). \tag{2.13}$$

It follows from (2.10), (2.12) and (2.13) that g solves the system of equations

$$U(g) = \frac{d}{dz}(V(g)), \quad S_\mu(V(g)) = T_\mu(g), \quad 0 \leq \mu \leq k-2. \tag{2.14}$$

We distinguish here two sub-cases.

Case 1A. Here we assume that the dimension of the solution space of (2.14) is 1, that is, every common solution of the equations (2.14) is a constant multiple of g.

Then (2.11) and a standard reduction procedure [10, p.126] give a first order equation

$$p_1 g' + p_0 g = 0, \quad p_1 \not\equiv 0,$$

with the p_j rational functions in the λ_ν and their derivatives, and it follows that

$$T(r, g'/g) = O(\log rT(r,F)) \quad (n.e.).$$

But then, since $F = -h/g$, (2.13) gives

$$T(r,F) = O(\log rT(r,F)) \quad (n.e.)$$

and F is a rational function, as asserted.

Case 1B. Here we assume that the system (2.14) has a solution G with G/g non-constant. (In particular this will be the case if the system (2.14) is trivial.)

Defining H by $H = V(G)$, we thus have, by (2.14),

$$H' = U(G), \quad S_\mu(H) = T_\mu(G), \quad 0 \le \mu \le k-2.$$

In particular the equations (2.10) hold with g and h replaced by G and H respectively, and so by Lemma 2.4 the functions $f_j H + f_j' G$ are solutions of (2.4). Hence there are polynomials g_j of degree at most $k-1$ such that

$$f_j H + f_j' G - g_j h - g_j' g = 0 \tag{2.15}$$

for $1 \le j \le k$.

We proceed almost verbatim as in [7] and regard the equations (2.15) as a system of k equations in H, G, h, g with rational coefficients f_j, f_j', g_j, g_j', and observe that the rank of the coefficient matrix is at most 3, since the system has a non-trivial solution. We assert that the rank is precisely 3. Assuming this not to be the case, there are rational functions ϕ_m for $1 \le m \le 3$, not all identically zero, as well as rational functions ψ_m, $1 \le m \le 3$, again not all identically zero, such that we have

$$\phi_1 f_j' + \phi_2 f_j = \phi_3 g_j, \quad \psi_1 f_j' + \psi_2 f_j = \psi_3 g_j'$$

for $1 \le j \le k$. Since $W(f_1, \ldots, f_k)$ is not identically zero, neither ϕ_3 nor ψ_3 can be identically zero, and we may therefore assume that $\phi_3 \equiv \psi_3 \equiv 1$. Thus

$$\phi_1 f_j'' + f_j'(\phi_1' + \phi_2 - \psi_1) + f_j(\phi_2' - \psi_2) = 0$$

for $1 \le j \le k$ whence, in view again of the fact that $W(f_1, \ldots, f_k) \not\equiv 0$, we must have

$$\phi_1 \equiv \phi_1' + \phi_2 - \psi_1 \equiv \phi_2' - \psi_2 \equiv 0,$$

which gives $g_j = \phi_2 f_j$. But then $W(g_1, \ldots, g_k) = (\phi_2)^k W(f_1, \ldots, f_k)$ so that ϕ_2 must be constant, since $f_1, \ldots, f_k$ and $g_1, \ldots, g_k$ are solutions of $w^{(k)} = 0$. Now, by (2.15), for $1 \le j \le k$,

$$f_j(H - \phi_2 h) + f_j'(G - \phi_2 g) = 0$$

and again, since $W(f_1, \ldots, f_k)$ is not identically zero, we must have $H = \phi_2 h$ and $G = \phi_2 g$, contradicting the assumption that G/g is non-constant.

Thus the rank of the system (2.15) is 3, and we may solve for $-F = h/g$ as a quotient of determinants in the f_j, g_j and their derivatives of first order. Hence F is a rational function.

Case 2. Here we assume that $c_{0,\mu} \equiv 0$ for $0 \leq \mu \leq k-1$ in (2.10).

In this case the equations (2.10) are obviously satisfied with g and h replaced by 0 and 1 respectively, and consequently so are the equations (2.9), so that by Lemma 2.4 the f_j are solutions of (2.4). Hence each A_q in (2.4) is identically zero, and we may write, for $1 \leq j \leq k$,

$$f_j h + f_j' g = g_j, \tag{2.16}$$

in which each g_j is a polynomial. Since $f_1 f_2' - f_1' f_2 \not\equiv 0$ we have

$$F = -h/g = (f_1' g_2 - f_2' g_1)/(f_1 g_2 - f_2 g_1),$$

so that again F is a rational function. □

Since F is a rational function, g is a polynomial, and by (2.5) so are the A_q. Moreover the w_j are polynomials and, since the w_j form a fundamental solution set of (2.4), the A_q must all vanish identically. Thus (2.12) gives

$$h' = -(k-1)g''/2, \quad h = -(k-1)g'/2 - c, \tag{2.17}$$

with c a constant, so that

$$F = \frac{(k-1)g'}{2g} + \frac{c}{g}, \tag{2.18}$$

holds, using (2.2). Since F is non-constant, so is g.

We assert that g has degree at most 2. To see this, recall that the w_j defined by (2.3) solve (2.4), with the A_q all identically zero. If g has degree greater than 2, it follows from (2.17) that w_k has degree at least $k+1$, and this is a contradiction. Thus g has degree at most 2, and it follows from (2.18) that F has the form (1.2) or (1.3).

Finally, suppose in the converse direction that F is given by (1.2) or (1.3). Then F has the form (2.18) with g a polynomial of degree at most 2. In this case we define f locally and h by

$$\frac{f'}{f} = F, \quad h = -\frac{(k-1)g'}{2} - c = -gF.$$

Define the f_j and w_j by (2.3). Then the w_j are polynomials, of degree at most $k-1$ since g is at most quadratic. Thus the w_j all solve $w^{(k)} = 0$ and we have (2.8), for some constant c_k, possibly 0. We then apply the same properties of Wronskians used in Lemma 2.3, but in reverse, to obtain locally (2.7) and

$$W(f_1, \dots, f_k, f) = c_k f g^{-k}.$$

If $c_k = 0$ then $f_1, \dots, f_k, f$ are linearly dependent and $\Psi_k(F) = f^{(k)}/f \equiv 0$. If $c_k \neq 0$ then $\Psi_k(F) = f^{(k)}/f$ is a constant multiple of g^{-k} and so is meromorphic without zeros.

Lemma 2.1 implies that if (ii) is satisfied, then $\Psi_k(F)$ has a pole and is thus nonconstant. On the other hand, if (ii) is not satisfied, then F has the form $F(z) = j/(z-a)$ if $\deg g = 1$ and, by (1.4), the form $F(z) = j/(z-a) + (k-1-j)/(z-b)$ if $\deg g = 2$, where $a, b \in \mathbb{C}$, $a \neq b$, and $j \in \{1, \dots, k-1\}$. Thus f is a polynomial of degree $k-1$ at most so that $\Psi_k(F) = f^{(k)}/f \equiv 0$.

3. Proof of Theorem 1.2

Let F be as in the statement of the theorem, and set $h(z) = z - 1/F(z)$. Since all zeros of F are simple by (i), we conclude that h has only simple poles. By (ii) we have $h'(a) \neq 0$ if a is a pole of F, and so h' has no zeros using (i).

If h is a rational function then h is Möbius, and this implies that F has the form stated. Suppose now that h is transcendental. Then by [18] (see also [3]) the order ρ of h is positive. Let $0 < \sigma < \rho$. By [13, Theorem 2] there are fixpoints z of h, with $|z|$ arbitrarily large, and with $|h'(z)| > |z|^\sigma$. These fixpoints must be simple poles of F, with

$$h'(z) = 1 - \frac{1}{\text{Res}(F, z)},$$

which contradicts (iii) and proves the theorem.

4. Proof of Theorem 1.3

The main tool is the following lemma of Pang and Zalcman; see [14, Lemma 2] and [15, Lemma 2].

Lemma 4.1. *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc, all of whose zeros have multiplicity at least k, and suppose that there exists $A \geq 1$ such that $|f^{(k)}(z)| \leq A$ whenever $f(z) = 0$, $f \in \mathcal{F}$. Then if $\mathcal{F}$ is not normal there exist, for each $0 \leq \alpha \leq k$, a number $r \in (0, 1)$, points $z_n \in D(0, r)$, functions $F_n \in \mathcal{F}$ and positive numbers ρ_n tending to zero such that*

$$\frac{F_n(z_n + \rho_n z)}{\rho_n^\alpha} \to F(z)$$

locally uniformly, where F is a nonconstant meromorphic function on $\mathbb{C}$ such that the spherical derivative $F^\#$ of F satisfies $F^\#(z) \leq F^\#(0) = kA + 1$ for all $z \in \mathbb{C}$.

Lemmas of this type have proved to be very useful in recent years; for a discussion we refer to a survey by Zalcman [20].

We shall need only the case $\alpha = k = 1$. Applying the lemma to the family of all functions $1/f$ with $f \in \mathcal{F}$ we obtain the following lemma.

Lemma 4.2. *Let $\mathcal{F}$ be a family of functions meromorphic on the unit disc. Suppose that there exists $\delta > 0$ such that if $f \in \mathcal{F}$ has a simple pole a, then $|\text{Res}(f, a)| \geq \delta$. Then if $\mathcal{F}$ is not normal, there exist a number $r \in (0, 1)$, points $z_n \in D(0, r)$, functions $F_n \in \mathcal{F}$ and positive numbers ρ_n tending to zero such that*

$$\rho_n F_n(z_n + \rho_n z) \to F(z)$$

locally uniformly, where F is a nonconstant meromorphic function on $\mathbb{C}$ such that $F^\#(z) \leq F^\#(0) = 1 + 1/\delta$ for all $z \in \mathbb{C}$.

Proof of Theorem 1.3. Without loss of generality we may assume that Ω is the unit disk. Suppose that $\mathcal{F}$ is not normal. Because of condition (ii) with $j = 0$ we can apply Lemma 4.2. Let r, z_n, F_n, ρ_n and F be as there so that

$$g_n(z) = \rho_n F_n(z_n + \rho_n z) \to F(z)$$

as $n \to \infty$.

Let a be a simple pole of F. Then, by Hurwitz's theorem, if n is sufficiently large, g_n has a simple pole a_n with $a_n \to a$. Since $z_n + \rho_n a_n$ is a simple pole of F_n with $\text{Res}(F_n, z_n + \rho_n a_n) = \text{Res}(g_n, a_n)$ we deduce from condition (ii) that

$|\mathrm{Res}(g_n, a_n) - j| \geq \delta$ for $j \in \{0, 1, \ldots, k-1\}$. This implies that $|\mathrm{Res}(F, a) - j| \geq \delta$ for $j \in \{0, 1, \ldots, k-1\}$. In particular, every pole of F is a pole of $\Psi_k(F)$, by Lemma 2.1.

Induction shows that $\Psi_k(g_n(z)) = \rho_n^k \Psi_k(F_n(z_n + \rho_n z))$. Thus $\Psi_k(g_n)$ has no zeros. If A is the set of poles of F then $\Psi_k(g_n) \to \Psi_k(F)$ locally uniformly on $\mathbb{C}\backslash A$, and either $\Psi_k(F) \equiv 0$ or $\Psi_k(F)$ has no zeros on $\mathbb{C}\backslash A$ by Hurwitz' theorem. In the latter case we deduce using the previous paragraph that $\Psi_k(F)$ has no zeros at all, and that $\Psi_k(g_n) \to \Psi_k(F)$ on the whole plane, by the maximum principle applied to $1/\Psi_k(g_n)$ and $1/\Psi_k(F)$.

Case 1. $\Psi_k(F)$ has no zeros.

It follows from Theorem 1.1 if $k \geq 3$ and from Theorem 1.2 if $k = 2$ that F has the form (1.2) or (1.3).

Suppose first that F has the form (1.3). Then $1/|\alpha| = |\mathrm{Res}(F, -\beta/\alpha)| \geq \delta$ so that $|\alpha| \leq 1/\delta$. On the other hand, $|\alpha| \geq |\alpha|/(1 + |\beta|^2) = F^{\#}(0) = 1 + 1/\delta$. This is a contradiction.

Suppose next that F has the form (1.2) but is not of the form (1.3). Then F has two poles, counting multiplicities. Choose $R > 0$ such that these poles are contained in $D(0, \delta R)$. Since F has no other poles we deduce from Hurwitz's theorem that if n is sufficiently large, then g_n has two poles in $D(0, \delta R)$, but no poles in $D(0, R) \setminus D(0, \delta R)$. Thus F_n has two poles in $D(z_n, \rho_n R)$, but no poles in $D(z_n, \rho_n R) \setminus D(z_n, \delta \rho_n R)$. From condition (iii) we deduce that

$$\left| \sum_{a \in D(0,\delta R)} \mathrm{Res}(g_n, a) - (k-1) \right| = \left| \sum_{a \in D(z_n, \delta\rho_n R)} \mathrm{Res}(F_n, a) - (k-1) \right| \geq \delta.$$

Thus

$$\left| \sum_{a \in D(0,\delta R)} \mathrm{Res}(F, a) - (k-1) \right| \geq \delta,$$

contradicting (1.4).

Case 2. $\Psi_k(F) \equiv 0$.

Since $|\mathrm{Res}(F, a) - j| \geq \delta$ for $j \in \{0, 1, \ldots, k-1\}$ if a is a simple pole of F, we deduce from Lemma 2.1 that F has no poles. Thus F is entire, and so is the function f defined by $f(z) = \exp(\int_0^z F(t)dt)$. Then $F = f'/f$ and thus $f^{(k)}/f = \Psi_k(F) \equiv 0$ by Lemma 1.1. Hence f is a polynomial. This implies that f is constant. Hence $F \equiv 0$, a contradiction. □

5. Remarks

5.1. While the statement of Theorem A makes no distinction between the cases $k = 2$ and $k \geq 3$, the proofs in [4] and [12] are quite different. The difference between Theorem 1.1 and Theorem 1.2 suggests that it may be difficult to treat the cases $k = 2$ and $k \geq 3$ with a uniform method.

5.2. For functions F of finite order the conclusion of Theorem 1.2 can also be obtained with the methods of [1]. In fact, Theorem 1.2 can be slightly strengthened for functions of finite order.

Theorem 5.1. *Let F be meromorphic, non-constant and of finite order in the plane, such that:*

(i) *all zeros of $\Psi_2(F) = F' + F^2$ are zeros or poles of F;*
(ii) *if a is a simple pole of F then $\mathrm{Res}(F, a) \neq 1$;*
(iii) *there exists $\delta > 0$ such that if a is a simple pole of F then $|\mathrm{Res}(F, a)| \geq \delta$.*

Then F has the form

$$F(z) = \frac{(z+c)^\ell}{(z+a)(z+c)^\ell + b} \tag{5.1}$$

with $a, b, c \in \mathbb{C}$, $b \neq 0$, $\ell \in \mathbb{N}$ or the form (1.3).

If, in addition, all zeros of F are simple, then F has the form (1.2) *with $k = 2$ or the form* (1.3).

Proof. Define $g = 1/F$ so that $g' = -F'/F^2$. By (ii) we have $g'(z) \neq 1$ if z is a pole of F, and we have $g'(z) = \infty \neq 1$ if z is a zero of F. Using (i) we see that $g'(z) \neq 1$ for all $z \in \mathbb{C}$. From (iii) we deduce that if $g(z) = 0$, then $|g'(z)| \leq 1/\delta$. Hence we can deduce from [1, Lemma 5] that g has the form

$$g(z) = z + a + \frac{b}{(z+c)^\ell},$$

with $a, b, c \in \mathbb{C}$, $b \neq 0$, $\ell \in \mathbb{N}$ or the form $g(z) = \alpha z + \beta$ with $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 1$. In the first case, F has the form (5.1) while in the second case, F has the form (1.3).

If all zeros of F are simple, then the form (5.1) is possible only for $\ell = 1$, in which case it reduces to (1.2) with $k = 2$. ☐

As our proof of Theorem 1.3 in the case $k = 2$ requires the conclusion of Theorem 1.2 only for functions of finite order, this approach suffices to obtain Theorem 1.3 in the case $k = 2$.

5.3. The hypothesis (ii) in Theorems 1.1 and 1.2 is satisfied not only when $F = f'/f$ where f is meromorphic without zeros, but also when the zeros of f have multiplicity at least k. This leads to the following corollary to these results.

Corollary 5.1. *Let f be meromorphic in $\mathbb{C}$ and $k \geq 2$. Suppose that all zeros of $ff^{(k)}$ are zeros of f of multiplicity at least k. Then f has the form $f(z) = e^{az+b}$, $f(z) = (az+b)^m$ or*

$$f(z) = a\frac{(z-b)^{n+k-1}}{(z-c)^n}, \tag{5.2}$$

where $a, b, c \in \mathbb{C}$, $a \neq 0$, $b \neq c$ and $n \in \mathbb{N}$, $m \in \mathbb{Z} \setminus \{0, \ldots, k-1\}$.

This result is probably known to researchers in the field, although for $k \geq 3$ it does not seem to have been stated explicitly before. For the case $k = 2$ it was stated in [13, Theorem 1, (ii)] that the only transcendental functions satisfying the hypothesis of Corollary 5.1 are those of the form $f(z) = e^{az+b}$.

Note that

$$\frac{d^k}{dz^k}\left(\frac{(z-b)^{n+k-1}}{(z-c)^n}\right) = (b-c)^k n(n+1)\ldots(n+k-1)\frac{(z-b)^{n-1}}{(z-c)^{n+k}},$$

which can be proved directly by induction, or using Lemma 1.1.

We also remark that if f has the form (5.2), then

$$\frac{f'(z)}{f(z)} = \frac{(k-1)z + nb - (n+k-1)c}{(z-b)(z-c)}.$$

Let $\mathcal{F}$ be the family of all functions f'/f where f has the form (5.2). Fixing $c = 0$ and letting $b \to 0$ we see that $\mathcal{F}$ fails to be normal. This shows that in Corollary 1.1 the condition that f and $f^{(k)}$ have no zeros cannot be replaced by the condition made in Corollary 5.1, namely that all zeros of $ff^{(k)}$ are zeros of f of multiplicity at least k.

5.4. We have already mentioned that Theorem A and Corollary 1.1 can be considered as analogous results according to Bloch's heuristic principle. To explain this in more detail, we fix $k \geq 2$ and say that a meromorphic function f has the property P if it is of the form $f = g'/g$ for some meromorphic function g such that g and $g^{(k)}$ have no zeros. By Lemma 1.1 this is equivalent to saying that f has the property P if all poles of f are simple, with negative integers as residues, and $\Psi_k(f)$ has no zeros. As pointed out in the introduction, Theorem A can be restated by saying that every function F meromorphic in the plane with property P is constant or of the form $F(z) = -n/(z+c)$. Similarly, Corollary 1.1 is equivalent to the statement that the family $\mathcal{F}$ of all functions meromorphic in some domain and having property P is normal.

Zalcman [19] originally introduced (a simplified version of) Lemma 4.1 in order to give a rigorous version of Bloch's heuristic principle. We note, however, that it does not seem possible to deduce Corollary 1.1 from Theorem A using only Lemma 4.2. In fact, assuming that $\mathcal{F}$ is not normal, we can can proceed as in the proof of Theorem 1.3 and use Lemma 4.2 to obtain functions F_n with property P and ρ_n, z_n such that $\rho_n F_n(z_n + \rho_n z) \to F(z)$ for some nonconstant function F meromorphic in the plane. As in the proof of Theorem 1.3 we find that $\Psi_k(F)$ has no zeros, that the residues at the poles of F are negative integers, and that F is not of the form $F(z) = -n/(z+c)$. However, F might have multiple poles and thus fail to have property P. Hence the above restatement of Theorem A is not applicable.

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