

Pairs of nonhomogeneous linear differential polynomials

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Abstract

Let f be transcendental and meromorphic in the plane and let the nonhomogeneous linear differential polynomials F and G be defined by

$$F = f^{(k)} + \sum_{j=0}^{k-1} a_j f^{(j)} + a, \quad G = f^{(n)} + \sum_{j=0}^{n-1} b_j f^{(j)} + b,$$

where $k, n \in \mathbb{N}$ and a, b and the a_j, b_j are rational functions. Under the assumption that F and G have few zeros it is shown that either F and G reduce to homogeneous linear differential polynomials in $f + c$, where c is a rational function which may be computed explicitly, or f has a representation as a rational function in solutions of certain associated linear differential equations, which again may be determined explicitly from the a_j, b_j and a and b .

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1 Introduction

This paper will use the standard notation of Nevanlinna theory [12], including $T(r, g)$ for the Nevanlinna characteristic of a function g meromorphic in the plane and $S(r, g)$ for any quantity with

$$S(r, g) = o(T(r, g)) \quad (\text{n.e.}),$$

where (n.e.) (“nearly everywhere”) means as $r \rightarrow \infty$ outside a set of finite measure. One of the many successes of the Nevanlinna theory has been its applicability to questions involving the value distribution of meromorphic functions and their derivatives. Hayman [11, 12] (see also [1]) proved that if f is meromorphic in the plane and f omits some finite value a while the k 'th derivative $f^{(k)}$ omits some finite non-zero value b , for some $k \geq 1$, then f is constant. The example $f(z) = e^z, a = b = 0$, shows that Hayman's result does not hold for $b = 0$, but the following theorem, proved in [3, 6, 15], deals with this exceptional case.

THEOREM 1.1 ([3, 6, 15]) *Suppose that f is meromorphic in the plane and that f and $f^{(k)}$ have finitely many zeros, for some $k \geq 2$. Then $f(z) = R(z)e^{P(z)}$, with R a rational function and P a polynomial. In particular, f has finite order and finitely many poles.*

A number of papers [2, 5, 13, 15, 16, 18] treat a more general problem in which $f^{(k)}$ is replaced by a linear differential polynomial $F = f^{(k)} + \sum_{j=0}^{k-1} A_j f^{(j)}$, with the coefficients A_j rational functions. This generalization was taken a step further in [7, 8], replacing both f and $f^{(k)}$ in Theorem 1.1 by linear differential polynomials. Let k, n be positive integers, and define linear differential operators L, M by

$$L = D^k + \sum_{j=0}^{k-1} a_j D^j, \quad M = D^n + \sum_{j=0}^{n-1} b_j D^j, \quad D = \frac{d}{dz}. \quad (1.1)$$

The following was proved in [8].

THEOREM 1.2 ([8]) *Let g be meromorphic and non-constant in the plane and let $F_1 = L(g)$ and $G_1 = M(g)$, with L, M as in (1.1) and with the a_j and b_j rational functions. Assume that*

$$\overline{N}(r, 1/F_1) + \overline{N}(r, 1/G_1) = O(\log^+ T(r, g'/g) + \log r) \quad (n.e.), \quad (1.2)$$

and that the equations

$$L(w) = 0, \quad M(w) = 0, \quad (1.3)$$

have no non-trivial common (local) solution.

Then g has finite order and finitely many zeros and g'/g has a representation

$$\frac{g'(z)}{g(z)} = V(z) + \frac{P_1(Q_1(z) + \log R_1(z))(Q_1'(z) + R_1'(z)/R_1(z))}{R_1(z)e^{Q_1(z)} - 1} \quad (1.4)$$

in which V and R_1 are rational functions and P_1 and Q_1 are polynomials, and at least one of P_1 and R_1 is constant.

There is no real loss of generality in assuming that the equations (1.3) have no common local solution other than the trivial solution $w = 0$, for otherwise a standard reduction procedure [14] (see also [8, Lemma D]) gives linear differential operators L^*, M^*, N , with coefficients which are rational functions, such that $L = L^* \circ N$, $M = M^* \circ N$, in which case F_1 and G_1 may be regarded as linear differential polynomials in the meromorphic function $N(g)$.

The assumption (1.2) is stronger than the standard $S(r, g)$ condition, but it should be noted that $T(r, g'/g)$, rather than $T(r, g)$, is the right comparison function in (1.2). This is because it is easy to construct meromorphic functions g with no zeros and with poles of large multiplicity so that $T(r, g'/g)$ is small compared to $T(r, g)$. For such functions g any zero of $L(g)$ will be a zero of $L(g)/g$, and the growth of $L(g)/g$ is controlled by that of g'/g and the coefficients a_j .

The aim of the present paper is to prove a nonhomogeneous version of Theorem 1.2. In order to state the result it is necessary to collect some standard facts concerning linear differential operators, which are summarised in the next lemma. Proofs may be found in [8, Lemma D and Lemma 1].

LEMMA 1.1 ([8, 14]) *Let k and n be positive integers and let the linear differential operators L and M be as in (1.1), with the coefficients a_j and b_j meromorphic on some plane domain Ω . Assume that the equations (1.3) have no non-trivial common (local) solution. Then there exist linear differential operators P, Q, U, V and Y , with coefficients which are rational functions in the a_j, b_j and their derivatives, such that*

$$P \circ L + Q \circ M = 1 \quad \text{and} \quad Y = U \circ L = V \circ M, \quad (1.5)$$

in which 1 denotes the identity operator. The operators P, Q, U, V, Y may be calculated explicitly from L and M , and U, V, Y have orders $n, k, n+k$ and leading terms D^n, D^k, D^{n+k} respectively, where $D = d/dz$. Finally, the (local) solution space of the equation $Y(w) = 0$ is the direct sum of the (local) solution spaces of the equations $L(w) = 0$ and $M(w) = 0$.

The following theorem will be proved.

THEOREM 1.3 *Let f be transcendental and meromorphic in the plane. Let k and n be positive integers, and let $a_0, \dots, a_{k-1}$ and $b_0, \dots, b_{n-1}$ and a and b be rational functions. Assume that the equations (1.3), with L, M as in (1.1), have no non-trivial common (local) solution, and that*

$$F = L(f) + a \quad \text{and} \quad G = M(f) + b \quad (1.6)$$

do not vanish identically, and finally that

$$N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{G}\right) = S(r, f). \quad (1.7)$$

Define functions c and g by

$$c = P(a) + Q(b), \quad g = f + c, \quad (1.8)$$

where the linear differential operators P and Q are as in Lemma 1.1. Let Ω be a non-empty simply connected domain on which a and b and the coefficients a_j, b_j are all analytic, and define on Ω linearly independent solutions $u_1, \dots, u_k$ of $L(w) = 0$,

linearly independent solutions $v_1, \dots, v_n$ of $M(w) = 0$, and solutions u, v of $L(w) = a, M(w) = b$ respectively.

Then c is a rational function and

$$P(F) + Q(G) = f + c = g, \quad (1.9)$$

and at least one of the following holds:

(a) F and G satisfy

$$F = L(g), \quad G = M(g); \quad (1.10)$$

(b) f has a representation $f = R(u_1, \dots, u_k, v_1, \dots, v_n, u, v)$, where R is a rational function in $k + n + 2$ variables.

Some additional comments concerning Theorem 1.3 are in order. The significance of the conclusion (a) is that in this case (1.10) shows that F and G reduce to homogeneous linear differential polynomials in g : in particular, if F and G have sufficiently few zeros, then f may be determined by applying Theorem 1.2 to g . An example satisfying both cases (a) and (b) is given by $f(z) = e^z + e^{2z} + 1$ and

$$F(z) = f'(z) - f(z) + 1 = e^{2z}, \quad G(z) = f'(z) - 2f(z) + 2 = -e^z.$$

On the other hand the example

$$f(z) = e^z + z, \quad F(z) = f'(z) + f(z) - (z+1) = 2e^z, \quad G(z) = f'(z) - f(z) = 1 - z,$$

satisfies (b) but not (a), and shows that f is not determined solely by the operators L and M . Finally, $f(z) = 1/(1 + e^z)$ has

$$F(z) = f'(z) + f(z) = \frac{1}{(1 + e^z)^2} \neq 0, \quad G(z) = f'(z) - f(z) + 1 = \frac{e^{2z}}{(1 + e^z)^2} \neq 0.$$

It would be interesting to know whether Theorem 1.3 holds with $N(r, 1/F)$ and $N(r, 1/G)$ replaced in (1.7) by $\overline{N}(r, 1/F)$ and $\overline{N}(r, 1/G)$, but the present method does not appear to give this. It is reasonable also to ask whether the result holds with a weaker assumption on the a_j, b_j and a and b , but it is pointed out in the proof that at least two steps would be in doubt if the coefficients were only small functions in the sense of Nevanlinna theory.

2 A lemma required for Theorem 1.3

The proof of Theorem 1.3 will require the following consequence of a lemma from [4, 9, 10, 17, 19]. The version in [19] suffices for the present application, but the proof in [4] is somewhat simpler.

LEMMA 2.1 *Let δ be a positive real number, and let h be transcendental and meromorphic in the plane. Let p be a positive integer and let $c_0, \dots, c_{p-1}$ and A be rational functions. Set*

$$H = Q_p(h) + A \quad \text{where} \quad Q_p = D^p + \sum_{j=0}^{p-1} c_j D^j \quad \text{and} \quad D = \frac{d}{dz}.$$

Then either (i)

$$p\overline{N}(r, h) \leq N\left(r, \frac{1}{H}\right) + (1 + \delta)N(r, h) + S(r, h), \quad (2.1)$$

or (ii) h has a representation

$$h = R(h_1, \dots, h_{p+1}), \quad (2.2)$$

where R is a rational function in $p + 1$ variables and each h_j is a (local) solution of

$$Q_p(w) = d_j A, \quad (2.3)$$

with d_j a constant.

Proof. Define a linear differential operator Q^* of order at most $p + 1$ and a function H_1 by

$$Q^* = Q_p \quad (\text{if } A \equiv 0), \quad Q^* = (D - A'/A) \circ Q_p \quad (\text{if } A \not\equiv 0), \quad H_1 = Q^*(h).$$

Then

$$\begin{aligned} H_1 &= Q_p(h) = H \quad (\text{if } A \equiv 0), \\ H_1 &= (Q_p(h))' - (A'/A)Q_p(h) = H' - (A'/A)H \quad (\text{if } A \not\equiv 0). \end{aligned} \quad (2.4)$$

In particular, H_1 is a homogeneous linear differential polynomial in h and by the result from [4, 19] there are two possibilities, the first of which is that h has a representation as a rational function in (local) solutions of the equation $Q^*(w) = 0$. Since every such local solution solves (2.3) for some constant d_j this gives (2.2) and conclusion (ii) of Lemma 2.1.

By the result from [4, 19] the second possibility is that

$$m\left(r, \frac{1}{H_1}\right) \leq m(r, H_1) + (2 + \delta)N(r, h) + S(r, h).$$

Since

$$m(r, H_1) \leq m(r, H) + S(r, h), \quad m\left(r, \frac{1}{H}\right) \leq m\left(r, \frac{1}{H_1}\right) + S(r, h),$$

using (2.4) and the lemma of the logarithmic derivative if $A \not\equiv 0$, this gives

$$m\left(r, \frac{1}{H}\right) \leq m(r, H) + (2 + \delta)N(r, h) + S(r, h). \quad (2.5)$$

Add $N(r, H) + N(r, 1/H)$ to both sides of (2.5). Since A and the coefficients of Q_p are rational functions,

$$N(r, H) = N(r, Q_p(h) + A) = N(r, h) + p\overline{N}(r, h) + S(r, f), \quad (2.6)$$

and applying Nevanlinna's first fundamental theorem to H leads at once to (2.1). $\square$

REMARK 2.1

The result in [4] shows that if $A \equiv 0$ then either (2.5) holds or h is a rational function in local solutions of $Q_p(w) = 0$, and this is proved under the weaker hypothesis that the coefficients c_j of the linear differential operator satisfy $T(r, c_j) = S(r, h)$. However Lemma 2.1 requires the relation (2.6), which may fail if the coefficients are only assumed to be small functions in the sense of Nevanlinna theory. For example, let q be a transcendental entire function such that $q'(z) = 0$ implies $q(z) \neq 0$ and

$$T(r, q''/q') = S(r, q), \quad T(r, q) = O(N(r, 1/q')) \quad (\text{n.e.}), \quad (2.7)$$

and define f by $f = q/q'$. Then $N(r, f) = N(r, 1/q') \neq S(r, f)$ and $T(r, q''/q') = S(r, f)$ but

$$f' + (q''/q')f = 1$$

has no poles. Such an entire function q may be constructed, for example, by writing

$$\frac{G'(z)}{G(z)} = \frac{2\pi iz^2}{e^{2\pi iz} - 1}, \quad q(z) = c + \int_0^z G(t)dt,$$

for some suitable constant c .

3 Proof of Theorem 1.3

Assume that f , F and G and the coefficients are as in the statement of Theorem 1.3. Let c , g and the linear differential operators P, Q, U, V, Y be defined as in (1.5) and (1.8). Then (1.9) follows at once from (1.5) and (1.6). Also (1.6) and (1.8) give

$$F = L(f) + a = L(g) + a - L(c), \quad G = M(f) + b = M(g) + b - M(c). \quad (3.1)$$

By (1.5) and (1.6),

$$U(F) = V(G) + d, \quad \text{where} \quad d = U(a) - V(b) \quad (3.2)$$

is a rational function.

LEMMA 3.1 *If either $U(F)$ or $V(G)$ is a rational function then f satisfies conclusion (b) of Theorem 1.3.*

Proof. Assume without loss of generality that $U(F)$ is a rational function. Then so is $V(G)$, by (3.2). If neither $U(F)$ nor $V(G)$ vanishes identically then each of F and G solves a nonhomogeneous linear differential equation with rational coefficients, so that (1.7) and the lemma of the logarithmic derivative give

$$T(r, F) + T(r, G) \leq m(r, 1/F) + m(r, 1/G) + S(r, f) = S(r, f),$$

which on combination with (1.8) and (1.9) leads to

$$T(r, f) \leq T(r, g) + S(r, f) = S(r, f),$$

an obvious contradiction.

Assume without loss of generality that $U(F) \equiv 0$. Then $0 = U(F) = Y(f) + U(a)$ by (1.5) and (1.6), so that with the u_j, v_j and u as defined in Theorem 1.3, the function $f + u$ solves $Y(w) = 0$ and is a linear combination of $u_1, \dots, u_k, v_1, \dots, v_n$ on Ω . Thus f satisfies conclusion (b) of Theorem 1.3. $\square$

Assume from now on that

$$U(F) \text{ and } V(G) \text{ are transcendental.} \quad (3.3)$$

It is obvious from (3.1) that if $a - L(c), b - M(c)$ both vanish identically, then (1.10) and conclusion (a) of Theorem 1.3 hold. Assume henceforth without loss of generality that

$$B = a - L(c) \not\equiv 0. \quad (3.4)$$

LEMMA 3.2 *The function f satisfies*

$$T(r, f) \leq N\left(r, \frac{1}{g}\right) + \overline{N}(r, f) + S(r, f). \quad (3.5)$$

Proof. Write

$$g = Bg^* \quad \text{where} \quad B = a - L(c) \not\equiv 0,$$

using (3.4). Then B is a rational function and (1.1), (3.1) and (3.4) give

$$F = L(g) + B = L(Bg^*) + B = B(L^*(g^*) + 1), \quad (3.6)$$

where L^* is a linear differential operator of order k with rational functions as coefficients. If $L^*(g^*)$ is constant, then F is a rational function, by (3.6), and so is $U(F)$, which contradicts (3.3). Hence $L^*(g^*)$ is non-constant and applying Milloux's inequality [12, p.57] to g^* and $L^*(g^*)$ in conjunction with (1.7) leads to (3.5). $\square$

The proof of Theorem 1.3 will now be divided into two cases.

Case 1: suppose that $d \not\equiv 0$ in (3.2).

Define linear differential operators U_1, V_1 by

$$U_1 = (D - d'/d) \circ U, \quad V_1 = (D - d'/d) \circ V, \quad D = d/dz. \quad (3.7)$$

Set

$$H = U_1(F) = V_1(G), \quad (3.8)$$

using (3.2) and (3.7). If $H \equiv 0$ then by (3.2), (3.7) and (3.8) there exist constants α, β such that $U(F) = \alpha d$ and $V(G) = \beta d$, which contradicts (3.3) since d is a rational function. Thus $H \not\equiv 0$. Set

$$\phi = \frac{gH}{FG} = \frac{P(F)V_1(G)}{FG} + \frac{Q(G)U_1(F)}{GF}, \quad (3.9)$$

using (1.9) and (3.8). Since P, Q, U_1, V_1 are linear differential operators with rational functions as coefficients, this gives

$$m(r, \phi) = S(r, f).$$

Consider next $N(r, \phi)$. Suppose that f has a pole of multiplicity m at z_0 , with z_0 large. Then g, H, F and G have poles at z_0 of multiplicities $m, m+n+k+1, m+k$ and $m+n$ respectively, so that ϕ has a simple pole at z_0 . But (3.9) shows that ϕ is a polynomial with rational functions as coefficients in the logarithmic derivatives $F^{(j)}/F$ and $G^{(j)}/G$ of F and G , each of which has poles of bounded multiplicity. Thus it follows using (1.7) that

$$T(r, \phi) \leq N(r, \phi) + S(r, f) \leq \overline{N}(r, f) + S(r, f). \quad (3.10)$$

Writing $1/gH = 1/\phi FG$ and using (1.7) and (3.10) now leads to

$$N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{H}\right) \leq N\left(r, \frac{1}{gH}\right) + O(\log r) \leq \overline{N}(r, f) + S(r, f). \quad (3.11)$$

Combining (3.5) and (3.11) gives

$$T(r, f) \leq 2\overline{N}(r, f) + S(r, f). \quad (3.12)$$

Let ε and δ be small and positive, with δ small compared to ε . By (1.5), (1.6), (3.7) and (3.8) the function H has a representation

$$H = ((D - d'/d) \circ U)(F) = ((D - d'/d) \circ Y)(f) + ((D - d'/d) \circ U)(a) \quad (3.13)$$

as a (possibly nonhomogeneous) linear differential polynomial in f , of order $k+n+1$, and with rational functions as coefficients. Lemma 2.1 now implies that there are two possibilities, the first of which is that

$$(k+n+1)\overline{N}(r, f) \leq N\left(r, \frac{1}{H}\right) + (1+\delta)N(r, f) + S(r, f), \quad (3.14)$$

which gives

$$(k+n+1)\overline{N}(r, f) \leq N\left(r, \frac{1}{H}\right) + (2+2\delta)\overline{N}(r, f) + S(r, f),$$

using (3.12). Since $k+n+1 \geq 3$, it follows that

$$\overline{N}(r, f) \leq (1+\varepsilon)N\left(r, \frac{1}{H}\right) + S(r, f),$$

which, using (3.11) twice, leads to

$$N\left(r, \frac{1}{g}\right) \leq \varepsilon N\left(r, \frac{1}{H}\right) + S(r, f) \leq \varepsilon \overline{N}(r, f) + S(r, f).$$

But combining this estimate with (3.5) gives

$$N(r, f) \leq T(r, f) \leq (1+\varepsilon)\overline{N}(r, f) + S(r, f), \quad (3.15)$$

so that applying (3.11) and (3.14) again leads this time to

$$\begin{aligned} (k+n+1)\overline{N}(r, f) &\leq N\left(r, \frac{1}{H}\right) + (1+\delta)N(r, f) + S(r, f) \\ &\leq (1+(1+\delta)(1+\varepsilon))\overline{N}(r, f) + S(r, f). \end{aligned}$$

Since $k+n+1 \geq 3$ and ε and δ are small, it follows that $\overline{N}(r, f) = S(r, f)$, contradicting (3.15).

This leaves as the only possibility flowing from Lemma 2.1 that f has a representation

$$f = R(y_1, \dots, y_{k+n+2}),$$

where R is a rational function in $k + n + 2$ variables and by (3.13) each y_j is, for some constant d_j , a solution on Ω of

$$((D - d'/d) \circ Y)(w) = d_j((D - d'/d) \circ U)(a).$$

But this gives, for some constant e_j , using (3.2),

$$Y(y_j) = d_j U(a) + e_j d = (d_j + e_j) U(a) - e_j V(b).$$

Hence, with u_j, v_j, u and v as defined in Theorem 1.3, the function $y_j - (d_j + e_j)u + e_j v$ solves $Y(w) = 0$ on Ω and is a linear combination of $u_1, \dots, u_k, v_1, \dots, v_n$, so that f satisfies conclusion (b). This completes the proof in Case 1. It remains to consider:

Case 2: suppose that $d \equiv 0$ in (3.2).

The proof in this case is somewhat simpler. This time H and ϕ are defined using (1.9) and (3.2) by

$$H = U(F) = V(G), \quad \phi = \frac{gH}{FG} = \frac{P(F)V(G)}{FG} + \frac{Q(G)U(F)}{GF}, \quad (3.16)$$

and $H \not\equiv 0$ by (3.3). The same analysis as in the proof of Case 1 gives $T(r, \phi) = S(r, f)$, so that (3.11) becomes

$$N\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{H}\right) = S(r, f). \quad (3.17)$$

Hence (3.12) is replaced using (3.5) by

$$T(r, f) \leq \overline{N}(r, f) + S(r, f). \quad (3.18)$$

Again let δ be small and positive. By (1.5), (1.6) and (3.16) the function H has this time a representation

$$H = U(F) = Y(f) + U(a) \quad (3.19)$$

as a (possibly nonhomogeneous) linear differential polynomial in f , of order $k + n$, and with rational functions as coefficients. Lemma 2.1 again gives two possibilities,

the first being that

$$\begin{aligned}(k+n)\overline{N}(r, f) &\leq N\left(r, \frac{1}{H}\right) + (1+\delta)N(r, f) + S(r, f) \\ &\leq (1+\delta)\overline{N}(r, f) + S(r, f),\end{aligned}$$

using (3.17) and (3.18). Since $k+n \geq 2$, it follows that

$$\overline{N}(r, f) = S(r, f),$$

which obviously contradicts (3.18).

By Lemma 2.1 this forces a representation

$$f = R(y_1, \dots, y_{k+n+1}),$$

where R is a rational function in $k+n+1$ variables and each y_j is, for some constant d_j , a local solution of $Y(w) = d_j U(a)$, so that $y_j - d_j u$ solves $Y(w) = 0$. Hence f again satisfies conclusion (b) of Theorem 1.3. This completes the proof of the theorem.

REMARK 3.1

The first inequality of (3.11) uses in an essential way the assumption that a, b and the coefficients a_j, b_j are rational functions. Were they only assumed to be small functions compared to f , then in principle H might have multiple zeros at multiple poles of g . Indeed, let $g = q^2/q'$, where q is an entire function as in (2.7), and set $H = (g' + (q''/q')g)'$. Then $T(r, q''/q') = S(r, g)$ but $H = 2q'$ and $gH = 2q^2$, so that H has zeros which are not zeros of gH .

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