The distribution of finite values of meromorphic functions with deficient poles

J.K. Langley

October 16, 2000

Abstract

We prove the existence of unbounded open subsets S of the complex plane with the following property. If f is a function transcendental and meromorphic in the plane, the poles of which have positive Nevanlinna deficiency, then f takes every finite value, with at most one exception, infinitely often in the complement of S.

A.M.S. Classification: 30D35.

1 Introduction

Let F be a family of functions meromorphic in the plane. A subset S of the plane is called a Picard set for F if every transcendental function f in F takes every element of the extended complex plane, with at most two exceptions, infinitely often in the complement of S.

For the class $M(\delta)$ of functions f meromorphic in the plane and with Nevanlinna deficiency

$$\delta(\infty, f) \geq \delta > 0, \tag{1}$$

the following was proved by Anderson and Clunie [1].

Theorem 1.1 ([1]) *Let $q > 1$ and $0 < \delta \leq 1$. Then there exists a positive constant K, depending only on q and δ, such that if the complex sequence (a_m) and the positive sequence (d_m) satisfy*

$$|a_{m+1}/a_m| > q \tag{2}$$

and

$$\log 1/d_m > K(\log |a_m|)^2 \tag{3}$$

for all m, then the set

$$S = \bigcup_{m=1}^{\infty} B(a_m, d_m) = \bigcup_{m=1}^{\infty} \{z : |z - a_m| < d_m\} \tag{4}$$

is a Picard set for $M(\delta)$.

Theorem 1.1 marks a strong departure from the situation for the class M of all functions meromorphic in the plane, for which there are no unbounded open Picard sets [5]. It was subsequently proved by Toppila [6] that Theorem 1.1 holds with a constant K depending only on q and not on δ. The condition (3) is essentially sharp, in that Toppila [6, Theorem 2] refined an example of Baker and Liverpool [2] by showing that for each $q > 1$ there exists a transcendental entire function f having all but finitely many of its zeros and 1-points in the union (4), with

$$a_m = (-1)^{m+1} q^m, \quad \log 1/d_m = \frac{(\log |a_m|)^2}{2 \log q}.$$

Further results on Picard sets for functions with deficient poles may be found in [7].

In the present paper we return to a theme considered in [4], the question of whether there exist unions S of discs as in (2), (3) and (4), with the property that every function f transcendental and meromorphic in the plane and satisfying (1) must take every finite value, with at most one exception, infinitely often in the complement of S. The existence of such sets S is suggested by the fact that such an f can have at most one finite Picard value, and a result on these lines was proved in [4] for functions having $\delta > 2/3$ in (1). We prove here the following theorem.

Theorem 1.2 *Let $0 < \varepsilon < 1/2$. Then there exists a positive constant K, depending only on ε, with the following property. Let (a_m) be a complex sequence converging to infinity, with*

$$|a_m - a_{m'}| > \varepsilon |a_m| \tag{5}$$

for $m \neq m'$, and let d_m satisfy (3). If f is a function transcendental and meromorphic in the plane and satisfying (1), then f takes every finite complex value, with at most one exception, infinitely often in the complement of the set (4).

The condition (5) seems slightly more natural than (2), and does not introduce major difficulties. The key difference, however, between Theorem 1.2 and the results of [1, 6, 7] is that just one finite exceptional value is allowed rather than two. This means, in particular, that the Schottky-type normal families methods used in [1, 6] cannot be applied to prove Theorem 1.2. Further, in contrast to [4], the deficiency in (1) is only required to be positive.

2 A lemma needed for Theorem 1.2

The following application of the maximum principle for subharmonic functions is a modification of the argument of [6, pp.181-2].

Lemma 2.1 *Let $0 < t < s < r$ and assume that*

$$s_j > 0, \quad t < |b_j| - s_j < |b_j| + s_j < s \tag{6}$$

for $j = 1, \ldots, M$. Set

$$\Omega = \{z : t < |z| < r\} \setminus \bigcup_{j=1}^{M} E_j, \tag{7}$$

in which E_j is the closed disk $\{z : |z - b_j| \leq s_j\}$. Let u be subharmonic and non-positive on Ω, and continuous on the closure of Ω, and let $v(z)$ be the Poisson integral

$$v(z) = \frac{1}{2\pi} \int_0^{2\pi} -u(re^{i\theta}) \left(\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \right) d\theta \tag{8}$$

of $-u$ in $B(0, r)$. Then for z in Ω we have

$$u(z) \leq -v(z) + C(z) m_0(r, -u) \leq \left(\left(\frac{|z| - r}{|z| + r} \right) + C(z) \right) m_0(r, -u), \tag{9}$$

in which

$$m_0(r,-u) = \frac{1}{2\pi}\int_0^{2\pi} -u(re^{i\theta})d\theta \geq 0$$

and

$$C(z) = \left(\frac{1+t/r}{1-t/r}\right)\frac{\log r/|z|}{\log r/t} + \left(\frac{1+s/r}{1-s/r}\right)\sum_{j=1}^{M}\frac{\log 2r/|z-b_j|}{\log 2r/s_j}. \tag{10}$$

Proof. Since $-u \geq 0$, Harnack's inequality gives

$$\left(\frac{r-|z|}{r+|z|}\right) m_0(r,-u) \leq v(z) \leq \left(\frac{r+|z|}{r-|z|}\right) m_0(r,-u) \tag{11}$$

on $B(0,r)$. Let

$$g(z) = \frac{\log r/|z|}{\log r/t}, \quad g_j(z) = \frac{\log 2r/|z-b_j|}{\log 2r/s_j}, \tag{12}$$

so that g and g_j are harmonic and non-negative on Ω. The inequalities (6) and (11) give

$$v(z) \leq \left(\frac{1+s/r}{1-s/r}\right) m_0(r,-u) \leq \left(\frac{1+s/r}{1-s/r}\right) m_0(r,-u)g_j(z) \tag{13}$$

for $|z-b_j| = s_j$, and

$$v(z) \leq \left(\frac{1+t/r}{1-t/r}\right) m_0(r,-u) \leq \left(\frac{1+t/r}{1-t/r}\right) m_0(r,-u)g(z) \tag{14}$$

for $|z| = t$. Thus (13) and (14) and the fact that $u \leq 0$ give

$$u(z) + v(z) \leq m_0(r,-u)\left(\left(\frac{1+t/r}{1-t/r}\right)g(z) + \left(\frac{1+s/r}{1-s/r}\right)\sum_{j=1}^{M} g_j(z)\right)$$

for z on the boundary of Ω and hence for all z in Ω, by the maximum principle. Combining this estimate with the left-hand inequality of (11) gives (9) and (10).

3 Proof of Theorem 1.2

Suppose that f is transcendental and meromorphic in the plane and satisfies (1), and that all but finitely many zeros and 1-points of f lie in the set (4), in which the sequence (a_m) converges to infinity such that (5) holds for some ε with $0 < \varepsilon < 1/2$, while d_m satisfies (3) for some $K > 0$. Denote by ε_1, B_j, C_j positive constants depending only on ε, with ε_1 small. Set

$$g = \frac{f-1}{f}, \quad \delta(1,g) \geq \delta > 0, \tag{15}$$

so that all but finitely many zeros and poles of g lie in the set (4).

Choose a positive sequence r_n such that, for each n,

$$er_n < r_{n+1} < e^{9/8}r_n, \quad \{z : e^{-4B_1}r_n < |z| < e^{4B_1}r_n\} \cap \bigcup_{m=1}^{\infty} B(a_m, d_m) = \emptyset. \tag{16}$$

For large n, take S_n with

$$e^{B_1}r_n < S_n < e^{2B_1}r_n \tag{17}$$

and such that

$$T(S_n',g) < B_2 T(S_n,g), \quad S_n' = S_n e^{1/T(S_n,g)}, \quad m(S_n,g'/g(g-1)) < B_3 \log(S_n T(S_n,g)). \tag{18}$$

Such S_n exist, by [3, p.38] applied to the function $\phi(s) = T(e^s, g)$.

We fix a large positive integer L, and assume that n is large compared to L. By (16),

$$n(S_n',g) + n(S_n',1/g) = n(r_n,g) + n(r_n,1/g) \leq B_4 T(S_n,g).$$

Thus a standard application of the differentiated Poisson-Jensen formula [3, p.22] in $B(0,S_n')$ gives

$$|g'(z)/g(z)| \leq T(S_n,g)^{B_5} \tag{19}$$

provided $B_6 \leq |z| \leq S_n$ and $|z - a_m| \geq 1$ for all m. Since

$$\frac{1}{g-1} = \frac{g'}{g(g-1)} \frac{g}{g'} \quad ,$$

(15) and (18) give

$$m(S_n, g/g') > (\delta/2) T(S_n,g). \tag{20}$$

We apply Lemma 2.1 to the function

$$u(z) = \log|g'(z)/g(z)| - B_5 \log T(S_n,g),$$

with $r = S_n$ and $t = r_{n-L}$, and with the $B(b_j,s_j)$ those discs $B(a_m,1)$ for which $t < |a_m| < r$. For z satisfying

$$r_{n-1} \leq |z| \leq r_n, \quad z \notin \bigcup_{m=1}^{\infty} B(a_m, \varepsilon_1 |a_m|), \tag{21}$$

Lemma 2.1 and (5), (16), (17), (19) and (20) lead to

$$\frac{|z|-r}{|z|+r} + C(z) \leq -B_7 + \frac{B_8}{L} + \frac{B_8 L}{\log 2S_n}$$

and

$$\log|g'(z)/g(z)| \leq -B_9 \delta T(S_n,g). \tag{22}$$

Recalling (15), this gives:

Lemma 3.1 *We have*

$$|\log g(z)| \leq \exp(-C_1 \delta T(r_n,g)), \quad \log|g(z)-1| \leq -C_2 \delta T(r_n,g) \tag{23}$$

for large n and for z satisfying (21).

From (15) and (23) we get

$$T(r_n,g) \leq C_3 \delta^{-1} m(r_{n-1}, 1/(g-1)) \leq C_4 \delta^{-1} T(r_{n-1},g), \tag{24}$$

and using (16), (23) and (24) we deduce at once:

Lemma 3.2 *There exists a positive real number ρ such that $T(r,g) < r^\rho$ for all large r. Further, if m is large then g has the same number of zeros as poles, counting multiplicity, in $B(a_m,d_m)$.*

We need next:

Lemma 3.3 *We have $g(z) = 1 + o(1)$ as $z \to \infty$ outside the union of the discs $B(a_m, \sqrt{d_m})$.*

Proof. Let m be large. By Lemma 3.1 it suffices to prove that $g(z) = 1 + o(1)$ for

$$\sqrt{d_m} \leq |z - a_m| \leq \varepsilon_1 |a_m|. \tag{25}$$

By Lemma 3.2 we may pair off the zeros and poles of g in $B(a_m, d_m)$ as α_ν and β_ν, with $1 \leq \nu \leq N$. We write

$$U(z) = g(z)P(z), \quad P(z) = \prod_{\nu=1}^{N} \left(\frac{z - \beta_\nu}{z - \alpha_\nu} \right), \tag{26}$$

so that U is analytic and non-zero in $|z - a_m| \leq \varepsilon_1 |a_m|$. For z satisfying (25) we have

$$|\log P(z)| \leq 2 \sum_{\nu=1}^{N} \left| \frac{\alpha_\nu - \beta_\nu}{z - \alpha_\nu} \right| \leq 8N\sqrt{d_m} \leq O(|a_m|^\rho)\sqrt{d_m} \leq d_m^{1/4}, \tag{27}$$

using (3) and Lemma 3.2. Combining (23) with (27) gives $\log U(z) = o(1)$ for $|z - a_m| = \varepsilon_1 |a_m|$, so that the lemma follows from the maximum principle and (27). Lemma 3.3 is proved.

Since Lemma 3.3 shows that $f(z)$ is large for large z outside the union of the discs $B(a_m, \sqrt{d_m})$, we may now essentially follow Toppila's proof in [6, pp.182-3].

Lemma 3.4 *Provided $K \geq K_0(\varepsilon)$ in (3), in which K_0 depends only on ε, we have*

$$T(r, f) = o(\log r)^2, \quad r \to \infty. \tag{28}$$

Proof. Let n be large, and apply Lemma 2.1 with $r = r_n$ and $t = r_{n'}$ satisfying

$$r^{1/100} \leq t \leq r^{1/50}, \tag{29}$$

and with the $B(b_j, s_j)$ those $B(a_m, \sqrt{d_m})$ for which $t < |a_m| < r$, the number of these being at most $C_5 \log r$. Choose $u(z) = -\log|f(z)|$ and let v be the Poisson integral of $-u$ as defined in Lemma 2.1. For $|z| = r_{n-1}$ we have, using (16),

$$\left(\frac{1 + t/r}{1 - t/r} \right) \frac{\log r/|z|}{\log r/t} \leq \frac{5}{4\log r}, \quad \log 2r/|z - b_j| \leq C_6$$

and so (3) and Lemma 2.1 give

$$u(z) \leq -v(z) + m(r, f) \left(\frac{5}{4\log r} + \frac{C_7 \log r}{K(\log t)^2} \right). \tag{30}$$

But v is harmonic in $B(0, r)$ and so

$$m_0(r_{n-1}, v) = v(0) = m(r, f), \quad -m(r_{n-1}, f) \leq m(r_n, f) \left(-1 + \frac{5}{4\log r_n} + \frac{10^4 C_7}{K \log r_n} \right),$$

using (29) and (30). If the constant K is large enough, we thus have, using (16) again,

$$\frac{m(r_n, f)}{m(r_{n-1}, f)} \leq 1 + \frac{3}{2\log r_n} \leq 1 + \frac{7}{4n}, \quad \log m(r_n, f) \leq O(1) + \frac{15}{8} \log n \leq O(1) + \frac{15}{8} \log\log r_n$$

for all large n, from which Lemma 3.4 follows, using (1).

We assume henceforth that K in (3) is sufficiently large that (28) holds.

Lemma 3.5 *Let $0 < \sigma < \varepsilon_1$ and let m be large. Then f has at least as many poles as zeros, counting multiplicity, in $B(a_m, \sigma|a_m|)$.*

Proof. Let $z_1, \ldots, z_p$ be the zeros of f in $B(a_m, \sigma|a_m|)$, and let $w_1, \ldots, w_q$ be the poles, in both cases repeated according to multiplicity. Set

$$h(z) = f(z) \prod_{\mu=1}^{p} (z - z_\mu)^{-1} \prod_{\nu=1}^{q} (z - w_\nu), \tag{31}$$

so that h is analytic and non-zero in $B(a_m, \sigma|a_m|)$. Using Lemma 3.4 we have

$$T(4|a_m|, h) \leq T(4|a_m|, f) + O(n(2|a_m|, f) + n(2|a_m|, 1/f)) \log |a_m| = o(\log |a_m|)^2$$

and a standard application of the Poisson-Jensen formula gives $\log |h(z)| = o(\log |a_m|^2)$ for z in $B(a_m, 1)$. Choose ζ with $\sqrt{d_m} \leq |\zeta - a_m| \leq 4\sqrt{d_m}$, and lying outside the union of the discs $B(w_\nu, \sqrt{d_m}/q)$. Then (28) and (31) give

$$0 \leq \log |f(\zeta)| \leq o(\log |a_m|)^2 + p \log 8\sqrt{d_m} + q(\log q - \log \sqrt{d_m}) \leq (p - q) \log \sqrt{d_m} + o(\log |a_m|)^2,$$

which on combination with (3) forces $p \leq q$. This proves Lemma 3.5.

Lemma 3.6 *For large n we have*

$$N(r_n, 1/f) \leq (1 + o(1))N(r_n, f). \tag{32}$$

Proof. By Lemma 3.3, f has infinitely many zeros. Applying Lemma 3.5 we see that if m is large and $|a_m| \leq r_n$ then to each zero z_μ of f in $B(a_m, d_m)$ corresponds a pole w_ν of f with

$$w_\nu = z_\mu(1 + o(1)), \quad \log r_n/|z_\mu| \leq \log r_n/|w_\nu| + o(1).$$

This gives, using (16),

$$N(r_n, 1/f) \leq N(r_n, f) + O(\log r_n) + o(n(e^{-4B_1} r_n, 1/f))$$

and (32) follows at once. Lemma 3.6 is proved.

We may now complete the proof of Theorem 1.2. Since $f(z)$ is large on $|z| = r_n$, by Lemma 3.3, we now have, using (1),

$$T(r_n, f) = N(r_n, 1/f) + O(1) \leq (1 + o(1))N(r_n, f) \leq (1 - \delta/2)T(r_n, f)$$

for large n, which is plainly impossible. This contradiction proves the theorem.

References

[1] J.M. Anderson and J. Clunie, Picard sets of entire and meromorphic functions, Ann. Acad. Sci. Fenn. Ser. A.I. 5 (1980), 27-43.

[2] I.N. Baker and L.S.O. Liverpool, Picard sets for entire functions, Math. Zeit. 126 (1972), 230-238.

[3] W.K. Hayman, Meromorphic functions, Oxford at the Clarendon Press, 1964.

[4] J.K. Langley, The distribution of finite values of meromorphic functions with few poles, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 12 (1985), 91-104.

[5] S. Toppila, Some remarks on the value distribution of entire functions, Ann. Acad. Sci. Fenn. Ser. A.I. No. 421 (1968), 11pp.

[6] S. Toppila, On the value distribution of meromorphic functions with a deficient value, Ann. Acad. Sci. Fenn. Ser. A.I. 5 (1980), 179-184.

[7] S. Toppila, Picard sets for meromorphic functions with a deficient value, Ann. Acad. Sci. Fenn. Ser. A.I. 5 (1980), 263-300.

School of Mathematical Sciences, University of Nottingham, NG7 2RD.