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ANALOGUES OF PICARD SETS FOR ENTIRE FUNCTIONS AND THEIR DERIVATIVES

by

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1. Introduction

A plane set E is a Picard set for entire functions if every transcendental entire function takes every finite complex value, with at most one exception, infinitely often in the complement of E. It is well known that certain uncountable sets are Picard sets for entire functions--perhaps that most striking result in this direction is the following improvement by Anderson and Clunie [3], and Toppila [9], of an earlier result of Baker and Liverpool [4]:

Theorem A

Suppose that $q > 1$ and $K > 0$ and that the complex sequence (a_n) and the positive sequence (ρ_n) satisfy, for all n,

$$|a_{n+1}| \geq q|a_n| \tag{1.1}$$

and

$$\log \frac{1}{\rho_n} \geq \frac{K}{\log q} (\log|a_n|)^2. \tag{1.2}$$

Then:

(a) if $K > \frac{1}{2}$, the union E of the discs $B(a_n,\rho_n) = \{z : |z - a_n| < \rho_n\}$ is a Picard set for entire functions:

(b) if $K \leq \frac{1}{2}$, there exists sequences (a_n) and (ρ_n) satisfying (1.1) and (1.2) such that the union E_1 of the discs $B(a_n,\rho_n)$ is not a Picard set for entire functions.

We shall be concerned with analogues of Theorem A for the value distribution of meromorphic functions and their derivatives. Specifically, if f is entire and transcendental, Hayman [6] proved that the equation $f^N(z)f'(z) = b$ must have infinitely many solutions, for any non-zero b, and integer N not less than two; Clunie [5] showed that the same conclusion holds for $N = 1$. The question then arises as to whether there are "exceptional" subsets of the plane outside which infinitely many of these points must lie, and in this direction, Anderson, Baker, and Clunie proved [2]:

Theorem B

Suppose that $a_n \to \infty$ such that, for all n,

$$\left|\frac{a_{n+1}}{a_n}\right| > q > 1.$$

Then if f is a transcendental, entire function, and N is an integer not less than two, the equation $f^N(z)f'(z) = b$ must have infinitely many solutions outside $E = \{a_n\}$, for any $b \neq 0$.

If (a_n) satisfies

$$\liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|\log|a_n|} > c > 0$$

the same conclusion holds for $N = 1$.

In the case where $N \geq 2$, we are able to extend Theorem B to obtain uncountable exceptional sets comparable to those of Theorem A, although our method fails for $N = 1$, due to difficulties in estimating the growth of f. We have

Theorem 1

Suppose that the complex sequence (a_n) and the positive sequence (ρ_n) satisfy, for all n,

$$\left|\frac{a_{n+1}}{a_n}\right| > q > 1$$

and

$$\log\frac{1}{\rho_n} > \frac{q^{\frac{1}{4}}+1}{q^{\frac{1}{4}}-1}\,\frac{8}{\log q}\,(\log|a_n|)^2.$$

Then, if f is a transcendental entire function, and N is an integer not less than two, the equation $f^N(z)f'(z) = b$ must have infinitely many solutions outside the union of the discs $B(a_n,\rho_n)$, for any $b \neq 0$.

Remark

This result was originally proved as part of the author's Ph.D. thesis, written under the supervision of I.N. Baker, to whom thanks are due.

We shall use the standard notation of Nevanlinna theory (see eg. [8]). If f is meromorphic in $|z| \leq r$, having $n(t,f)$ poles in $|z| \leq t$ (counting poles according to multiplicity), set

$$N(r,f) = \int_0^r (n(t,f) - n(0,f))\frac{dt}{t} + n(0,f)\log r$$

and

$$T(r,f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|d\theta + N(r,f) = m(r,f) + N(r,f)$$

where $\log^+ x = \max\{\log x, 0\}$. For f analytic in $|z| \leq r$, we shall require

$$M(r,f) = \max\{|f(z)| : |z| = r\}$$

and the estimate

$$\log^+ M(s,f) \leq \frac{r+s}{r-s} T(r,f)$$

for $0 < s < r$. We shall use the term "nearly everywhere" (n.e.) as follows: "n.e. as $r \to \infty$" will denote "as $r \to \infty$ outside a set of finite measure", while if R is large, and $S > R$, "n.e. in $R \leq r < S$" will denote "for all r in $R < r < S$ outside a set of small measure". We shall denote by $S(r,f)$ any quantity satisfying $S(r,f) = o(T(r,f))$ n.e. as $r \to \infty$.

A key role will be played by the following result ([8], p. 57):

Theorem C

Suppose that $f(z)$ is meromorphic and non-constant in the plane, and that

$$\psi(z) = \sum_{i=0}^{k} a_i(z) f^{(i)}(z)$$

is non-constant, where $k \geq 1$ and, for each i,

$$T(r,a_i) = S(r,f).$$

Then

$$T(r,f) < \bar{N}(r,f) + N(r,\frac{1}{f}) + \bar{N}(r,\frac{1}{\psi - 1}) - N_0(r,\frac{1}{\psi'}) + S(r,f)$$

where $N_0(r,\frac{1}{\psi'})$ counts only zeros of ψ' which are not roots of $\psi(z) = 1$, and $\bar{N}(r,f)$, $\bar{N}(r,\frac{1}{\psi - 1})$ count only points at which $f = \infty$, $\psi = 1$ respectively without regard to multiplicity.

We shall require the following result of Hayman [7]:

Theorem D

Suppose that $f(z)$ is entire and satisfies

$$T(r,f) = O(\log r)^2.$$

Then

$$\log|f(re^{i\theta})| \sim \log M(r,f)$$

as $z = re^{i\theta}$ tends to infinity outside an ε-set surrounding the zeros of f.

The term "ε-set" denotes a countable set of discs not meeting the origin, which subtend angles at the origin whose sum is finite.

2. Preliminary Lemmas

Lemma 1

Suppose that $P(z)$ is a polynomial of degree k with all its zeros $b_1,\ldots,b_k$ lying in $|z| < R_0$. Then, for $|z| = R \geq R_0$,

$$\left|\frac{P'(z)}{P(z)}\right| > \frac{k}{2R}.$$

Proof

We have

$$\frac{P'(z)}{P(z)} = \sum_{i=1}^{k} \frac{1}{z - b_i} = \frac{1}{z}\sum_{i=1}^{k} \left(1 - \frac{b_i}{z}\right)^{-1}$$

Now, if $|\omega| < 1$, $\mathrm{Re}\,\frac{1}{1-\omega} > \frac{1}{2}$, since $t = (1-\omega)^{-1}$ maps the unit disc $|\omega| < 1$ into the half-plane $\mathrm{Re}(t) > \frac{1}{2}$. Thus, if $|z| = R \geq R_0$, then $\left|\frac{b_i}{z}\right| < 1$ and so

$$\left|\frac{P'(z)}{P(z)}\right| = \frac{1}{R}\left|\sum_{i=1}^{k}\left(1 - \frac{b_i}{z}\right)^{-1}\right|$$

$$\geq \frac{1}{R}\left|\mathrm{Re}\sum_{i=1}^{k}\left(1 - \frac{b_i}{z}\right)^{-1}\right|$$

$$> \frac{k}{2R}.$$

Lemma 2

Suppose that $h(z)$ is regular and non-zero in $|z| \leq R$, and that $|\log|h(z)|| \leq M$ on $|z| = R$. Then for $|z| = r < R$,

$$\left|\frac{h'(z)}{h(z)}\right| \leq \frac{2MR}{(R-r)^2}$$

Proof

We have (see eg. [8], p. 22)

$$\frac{h'(z)}{h(z)} = \frac{1}{\pi}\int_0^{2\pi} \log|h(Re^{i\phi})|\frac{Re^{i\phi}}{(Re^{i\phi}-z)^2}\,d\phi$$

and simple estimates yield Lemma 2.

3. Proof of Theorem 1

Suppose that there exists a transcendental entire function $f(z)$, having only finitely many solutions of $f^N(z)f'(z) = b$ outside the union of the discs $D_n = B(a_n, \rho_n)$ such that,

$$\frac{a_{n+1}}{a_n} > q > 1 \tag{3.1}$$

and

$$\log\frac{1}{\rho_n} > \frac{q^{\frac{1}{4}}+1}{q^{\frac{1}{4}}-1}\,\frac{8}{\log q}\,(\log|a_n|)^2 \tag{3.2}$$

while N is an integer not less than two, and $b \neq 0$. It is clear that we may assume that $b = 1$, since otherwise we need only consider $g(z) = \alpha f(z)$ where $\alpha^{N+1} = b^{-1}$. We shall show by a series of steps that f is of very small growth, and finally obtain a contradiction. We set

$$F(z) = \frac{1}{N+1} f^{N+1}(z), \tag{3.3}$$

and note that all large 1-points of F' lie in the discs D_n. Applying Theorem C (with $\psi = F'$), we have

$$T(r,F) < N(r, \frac{1}{F}) + \bar{N}(r, \frac{1}{F'-1})$$

$$- N_0(r, \frac{1}{F''}) + S(r,F)$$

where $N_0(r, \frac{1}{F''})$ counts only zeros of $F''(z)$ which are not 1-points of $F'(z)$, and $S(r,F) = o(T(r,F))$ n.e. as $r \to \infty$. Hence

$$T(r,F) < N(r, \frac{1}{F}) + N(r, \frac{1}{F'-1})$$

$$- N(r, \frac{1}{F''}) + S(r,F) \tag{3.4}$$

Now, zeros of F are zeros of f, and thus have multiplicity at least 3, from (3.3). Thus, if $\hat{N}(r, \frac{1}{F''})$ counts the zeros of F'' which are not zeros of F, and which lie in the discs D_n, we have

$$N(r, \frac{1}{F}) - N(r, \frac{1}{F''}) \leq 2\,\bar{N}(r, \frac{1}{F}) - \hat{N}(r, \frac{1}{F''}) , \tag{3.5}$$

where $\bar{N}(r, \frac{1}{F})$ counts the points at which $F = 0$, without regard to multiplicity.

But then

$$\bar{N}(r, \frac{1}{F}) \leq \frac{1}{N+1} N(r, \frac{1}{F}) \leq \frac{1}{3} N(r,F) + O(1)$$

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and so (3.4) and (3.5) yield

$$T(r,F) < 4\left(N(\ , \frac{1}{F'-1}) - \hat{N}(r, \frac{1}{F''})\right) \tag{3.6}$$

n.e. as $r \to \infty$.

We define sequences p_n, t_n, and v_n as follows. In both cases counting multiplicities, let p_n be the number of 1-points of F' in the disc D_n, and t_n the number of zeros of F'' in D_n which are not zeros of F, and set

$$v_n = p_n - t_n. \tag{3.7}$$

Then for large M, we have n.e. in $q^{\frac{1}{2}}|a_M| \leq r \leq |a_{M+1}| - 1$,

$$\frac{1}{4} T(r,F) < O(\log r) + \sum_{n=1}^{M} p_n\left(\log \frac{r}{|a_n|} + o(1)\right)$$

$$- \sum_{n=1}^{M} t_n\left(\log \frac{r}{|a_n|} + o(1)\right). \tag{3.8}$$

Now,

$$\sum_{n=1}^{M} (p_n + t_n) \leq n(|a_M|+1, \frac{1}{F'-1}) + n(|a_M|+1, \frac{1}{F''})$$

$$= O(T(r,F) + S(r,F)) \tag{3.9}$$

for $r \geq q^{\frac{1}{2}}|a_M|$. Since F is transcendental, by the assumption on f, we have, from (3.8) and (3.9),

$$T(r,F) < \frac{9}{2} \sum_{n=1}^{M} v_n \log \frac{r}{|a_n|} \tag{3.10}$$

n.e. in $q^{\frac{1}{2}}|a_M| \leq r \leq |a_{M+1}| - 1$, if M is large enough. Now, whether or not the sequence (v_n) is bounded above, there must exist m_0 and infinitely many M such that

$$v_M = \max\{v_m : m_0 < m \leq M\} \geq 1 \tag{3.11}$$

since otherwise F would not be transcendental.

Suppose now that M is large, and satisfies (3.11). Then n.e. in $q^{\frac{1}{2}}|a_M| \leq r \leq |a_{M+1}| - 1$ we have, from (3.10),

$$T(r,F) < O(\log r) + \frac{9}{2} v_M \sum_{m=m_0}^{M} \log \frac{r}{|a_m|}. \tag{3.12}$$

But

$$\sum_{m=1}^{M} \log \frac{r}{|a_m|} < M \log \frac{r}{|a_1|}$$

$$= M(1 + o(1))\log r$$

$$\leq (1 + o(1))(\log q)^{-1}(\log r)^2 \tag{3.13}$$

for M and r as above, since

$$\log r \geq \log q^{\frac{1}{2}}|a_M| \geq (M - \tfrac{1}{2})\log q + \log|a_1|.$$

Thus, from (3.12),

$$T(r,F) < \frac{5}{\log q} v_M(\log r)^2 \tag{3.14}$$

n.e. in $q^{\frac{1}{2}}|a_M| \leq r \leq |a_{M+1}| - 1$, for any large M satisfying (3.11).

We go on to show that if M is large, and satisfies (3.11), then F has no zeros in D_M, and $v_M \leq 1$. From (3.14).

$$T(r,F' - 1) = T(r,F') + O(1)$$

$$< (1 + o(1))T(r,F)$$

$$< \frac{6}{\log q} v_M(\log r)^2 \tag{3.15}$$

n.e. in $q^{\frac{1}{2}}|a_M| \leq r \leq |a_{M+1}| - 1$. In particular, if M is large and satisfies (3.11),

$$T(q^{\frac{1}{2}}|a_M|, F' - 1) < \frac{7}{\log q} v_M(\log|a_M|)^2 \tag{3.16}$$

since (3.14) holds for some r with $q^{1/2}|a_M| \leq r \leq q^{3/4}|a_M|$. Now suppose that the 1-points of F' in D_M are $c_1, \ldots, c_{p_M}$. We set

$$P(z) = \prod_{k=1}^{p_M} (z - c_k) \tag{3.17}$$

and

$$F'(z) - 1 = h(z)P(z) \tag{3.18}$$

On the circle $|z - a_M| = 4$,

$$\log|P(z)| > p_M \log 3$$

(noting that $p_M \geq v_M \geq 1$) while

$$\log|F'(z) - 1| \leq \log M(|a_M| + 4, F' - 1)$$
$$\leq \frac{q^{\frac{1}{4}} + 1}{q^{\frac{1}{4}} - 1} T(q^{\frac{1}{2}}|a_M|, F' - 1). \qquad (3.19)$$

Thus, on $|z - a_M| = 4$, from (3.19),

$$\log|h(z)| < \frac{q^{\frac{1}{4}} + 1}{q^{\frac{1}{4}} - 1} T(q^{\frac{1}{2}}|a_M|, F' - 1)$$
$$- p_M \log 3$$

and the same estimate holds in $|z - a_M| < 4$, by the maximum principle. But for z in the disc D_M, we have $|z - c_k| < 2\rho_M$, and thus using (3.16),

$$\log|F'(z) - 1| = \log|P(z)| + \log|h(z)|$$
$$< p_M \log 2 - p_M \log 3$$
$$+ \frac{q^{\frac{1}{4}} + 1}{q^{\frac{1}{4}} - 1} \frac{7}{\log q} v_M(\log|a_M|)^2$$
$$< p_M\left(\frac{q^{\frac{1}{4}} + 1}{q^{\frac{1}{4}} - 1} \frac{8}{\log q} (\log|a_M|)^2\right)$$
$$- p_M \log \frac{1}{\rho_M}$$
$$< 0,$$

by (3.2), provided M is large enough. In particular, $F'(z) \neq 0$ in D_M. Since zeros of F are zeros of F', we conclude that F cannot have any zeros in D_M, if M is large and satisfies (3.11).

To show that $v_M \leq 1$, we note that, if M is large enough,

$$T(q^{\frac{1}{2}}|a_M|, h) \leq T(q^{\frac{1}{2}}|a_M|, F' - 1) + T(q^{\frac{1}{2}}|a_M|, \frac{1}{P})$$
$$\leq T(q^{\frac{1}{2}}|a_M|, F' - 1) + T(q^{\frac{1}{2}}|a_M|, P)$$
$$\leq \frac{7}{\log q} v_M(\log|a_M|)^2 + p_M \log 2q^{\frac{1}{2}}|a_M|$$
$$\leq \frac{8}{\log q} p_M(\log a_M)^2 \qquad (3.20)$$

using (3.16), and noting that $|P(0)| > 1$.

We estimate $\left|\log|h(z)|\right|$ for $z = re^{i\theta}$ lying in $|z - a_M| \leq 4$.

The Poisson-Jensen formula, applied to h in $|\omega| < R = q^{\frac{1}{2}}|a_M|$, yields

$$\log|h(z)| = \frac{1}{2\pi}\int_0^{2\pi} \log|h(Re^{i\theta})| \frac{(R^2 - r^2)\,d\phi}{R^2 + r^2 - 2Rr\cos(\theta - \phi)}$$

$$- \sum_{\substack{\xi = \text{zeros of} \\ h \text{ in } |\omega| < R}} \log\left|\frac{R^2 - \bar{\xi}z}{R(\xi - z)}\right| \qquad (3.21)$$

Now, any zero, ξ say, of h in $|\omega| < R$ is a zero of $F' - 1$, and thus $|\xi| < |a_{M-1}| + 1$, and so $|\xi - z| \geq \frac{1}{2}(1 - \frac{1}{q})|a_M|$, provided M is large enough. Thus (3.12) yields

$$\left|\log|h(z)|\right| \leq \frac{R + r}{R - r}\left(m(r,h) + m(R, \frac{1}{h})\right)$$

$$+ n\left(|a_{M-1}| + 1, \frac{1}{F' - 1}\right) \log \frac{4R}{(1 - \frac{1}{q})|a_M|}$$

$$\leq \hat{c}_1 (m(R,h) + m(R, \frac{1}{h}))$$

$$+ \hat{c}_2\, T(R,F' - 1) \qquad (3.22)$$

where we use $\hat{c}_1, \hat{c}_2, \hat{c}_3, \ldots$ to denote constants depending only on q. But

$$m(R, \frac{1}{h}) \leq m(R, \frac{1}{F' - 1}) + m(R,P)$$

$$\leq T(R,F' - 1) + p_M \log 2R + O(1)$$

and thus (recalling that $R = q^{\frac{1}{2}}|a_M|$) we have, from (3.16), (3.20) and (3.22)

$$\left|\log|h(z)|\right| \leq \hat{c}_3\, p_M(\log|a_M|)^2$$

for $|z - a_M| \leq 4$. Thus by Lemma 2,

$$\left|\frac{h'(z)}{h(z)}\right| \leq \hat{c}_4\, p_M(\log|a_M|)^2$$

for $|z - a_M| \leq 2$. But P(z) has all its zeros in the disc D_M, and so, by Lemma 1,

$$\left|\frac{P'(z)}{P(z)}\right| > \frac{2}{\rho_M}\, p_M$$

on $|z - a_M| = \rho_M$, and hence, using (3.2)

$$|P'(z)h(z)| > |P(z)h'(z)|$$

on $|z - a_M| = \rho_M$, if M is large enough. But, from (3.18),

$$F''(z) = P'(z)h(z) + P(z)h'(z)$$

and so, by Rouché's Theorem, F" has the same number of zeros in $|z - a_M| < \rho_M$ as $P'(z)h(z)$. But $h(z)$ does not vanish in D_M, while $P'(z)$ has $p_M - 1$ zero, which all lie in the convex hull of the set of zeros of $P(z)$ ([1], p. 29) and hence all lie in D_M. So F" has $p_M - 1$ zeros in D_M, and since, as we saw earlier, F does not vanish in D_M, we have

$$t_M = p_M - 1$$

for all large enough M satisfying (3.11); hence, for such M,

$$v_M = p_M - t_M = 1$$

and thus $v_n \le 1$ for all large n.

Returning to the estimate (3.10), we now have

$$T(r,F) < \frac{9}{2} \sum_{n=1}^{M} \log \frac{r}{|a_n|}$$

and hence, using (3.13),

$$T(r,F) < \frac{5}{\log q} (\log r)^2 \tag{3.23}$$

n.e. in $q^{\frac{1}{2}}|a_M| \le r \le |a_{M+1}| - 1$, if M is large enough, and hence

$$T(r,F) = O(\log r)^2$$

for all large r; in particular, F has order zero, and so (3.23) holds for $r = q^{\frac{1}{2}}|a_M|$, if M is large. Moreover,

$$\begin{aligned} T(r,F' - 1) &\le (1 + o(1))T(r,F) \\ &= O(\log r)^2 \end{aligned} \tag{3.24}$$

and, for large M,

$$\begin{aligned} T(q^{\frac{1}{2}}|a_M|, F' - 1) &< (1 + o(1)) \frac{5}{\log q} (\log q^{\frac{1}{2}}|a_M|)^2 \\ &< \frac{6}{\log q} (\log|a_M|)^2. \end{aligned} \tag{3.25}$$

We are now in a position to obtain a contradiction. We take a positive ε so small that the discs $B(a_n, 4\varepsilon|a_n|)$ are disjoint. Since all large zeros of $F' - 1$ lie in the discs $D_n = B(a_n, \rho_n)$ we see from Theorem D that

$|F'(z) - 1| > 10$, say, for large z outside the union of the discs $B(a_n, \varepsilon|a_n|)$. Now consider a large zero, z_0, of $f(z)$ (note that since F has order zero but is assumed transcendental, F and hence f must have infinitely many zeros). Since $F'(z) = f^N(z)f'(z)$ we have $F'(z_0) = 0$, and hence z_0 must lie in some disc $B(a_M, \varepsilon|a_M|)$; but then, by Rouche's Theorem, F' must have a 1-point, ω_0 say, in $B(a_M, \varepsilon|a_M|)$, which must lie in the smaller disc D_M.

We show now that

$$\log|F'(z) - 1| < 0 \text{ in } D_M. \tag{3.26}$$

On the circle $|z - a_M| = 4$, we have $|z - \omega_0| > 3$, while

$$\log|F'(z) - 1| \le \log M(|a_M| + 4, F' - 1)$$

$$< \frac{q^{\frac{1}{4}} + 1}{q^{\frac{1}{4}} - 1} \frac{6}{\log q} (\log|a_M|)^2$$

using (3.25). But then, by the maximum principle,

$$\log\left|\frac{F'(z) - 1}{z - \omega_0}\right| < \frac{q^{\frac{1}{4}} + 1}{q^{\frac{1}{4}} - 1} \frac{6}{\log q} (\log|a_M|)^2 - \log 3$$

for $|z - a_M| \le 4$, and so, for $|z - a_M| < \rho_M$,

$$\log|F'(z) - 1| < \frac{q^{\frac{1}{4}} + 1}{q^{\frac{1}{4}} - 1} \frac{6}{\log q} (\log|a_M|)^2 - \log 3 + \log(2\rho_M) < 0$$

using (3.2). Thus (3.26) holds, and $\text{Re}(F'(z) > 0$ in D_M.

In particular, z_0 cannot lie in D_M, since $F'(z_0) = 0$; however, z_0 lies in a component of $\{z : |F'(z)| < 1\}$ whose boundary is a level curve Γ, contained entirely in $B(a_M, \varepsilon|a_M|)$, on which $|F'(z)| = 1$. Since Γ must close in $B(a_M, \varepsilon|a_M|)$, Γ must pass through a 1-point of $F'(z)$ and hence must meet D_M. Consider now a ray from z_0 which does not meet the disc D_M, and suppose that the first point of intersection of this ray with Γ is at z_1, say. Starting from z_1 we may pass along Γ in either direction until the first point of intersection with the closed disc $\bar{D}_M$, and suppose that these points are

z_2, z_3, say. Then we may form a closed curve J, consisting of the line-segment from z_2 to z_3 and the arc of Γ which contains z_1, z_2, and z_3 and in whose interior lies z_0. We consider arg $F'(z)$ on this composite curve J.

On the open line-segment from z_2 to z_3 (which is contained in D_M) we have $Re(F'(z)) > 0$ and so arg $F'(z)$ cannot change by more than π as z moves from z_2 to z_3 along this line-segment. On the arc of Γ which contains z_1, z_2, and z_3 (and which forms the other component of J) we have $|F'(z)| = 1$ but this arc passes through no 1-points of $F'(z)$. Hence arg $F'(z)$ cannot change by more than 2π as z moves from z_3 to z_2 along this arc. We conclude that arg $F'(z)$ cannot increase by more than 3π as z describes the closed curve J once (in either direction) and hence by the argument principle $F'(z)$ can have at most one zero in the interior of the curve J. But z_0 lies in this interior, and is a zero of $F'(z)$ of multiplicity at least N, i.e. at least 2. Thus we have obtained a contradiction and conclude that $f(z)$ cannot be transcendental.

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