

Slowly growing meromorphic functions and the zeros of differences

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Abstract

Let f be a function transcendental and meromorphic in the plane with

$$\liminf_{r \rightarrow \infty} \frac{T(r, f)}{(\log r)^2} = 0.$$

Let $q \in \mathbb{C}$ with $|q| > 1$. It is shown that at least one of the functions

$$F(z) = f(qz) - f(z), \quad G(z) = \frac{F(z)}{f(z)}$$

has infinitely many zeros. This result is sharp. **MSC 2000:** 30D35.

1 Introduction

For a function f which is transcendental and meromorphic in the plane the forward differences $\Delta^n f$ are defined by [15, p.52]

$$\Delta f(z) = f(z+1) - f(z), \quad \Delta^{n+1} f(z) = \Delta^n f(z+1) - \Delta^n f(z), \quad n = 1, 2, \dots \quad (1.1)$$

There has been substantial recent interest [1, 3, 7, 8, 10, 12] in connections between the Nevanlinna theory and the difference operator, as well as meromorphic solutions of difference and functional equations. The papers [2, 14] investigated the zeros of the forward differences $\Delta^n f$ as defined in (1.1) and the divided differences $(\Delta^n f)/f$, partly in analogy with the following sharp theorem for derivatives [4, 6, 11], which uses notation from [9].

Theorem 1.1 ([4, 6, 11]) *Let f be transcendental and meromorphic in the plane with*

$$\liminf_{r \rightarrow \infty} \frac{T(r, f)}{r} = 0.$$

Then f' has infinitely many zeros. If, in addition, f is entire, then f'/f has infinitely many zeros.

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In the light of Theorem 1.1 the natural conjecture was advanced in [2] that if f is a transcendental meromorphic function of order less than 1 then Δf has infinitely many zeros, and if f is entire then so has $(\Delta f)/f$. The following theorem from [14] extended results from [2].

Theorem 1.2 ([14]) *Let $n \in \mathbb{N}$. Let f be transcendental and meromorphic of order $\rho < 1$ in the plane and assume that*

$$H_n(z) = \frac{\Delta^n f(z)}{f(z)}$$

is transcendental.

(i) *If H_n has lower order $\mu < \alpha < 1/2$, which holds in particular if $\rho < 1/2$, then*

$$\delta(0, H_n) \leq 1 - \cos \pi \alpha \quad \text{or} \quad \delta(\infty, f) \leq \frac{\mu}{\alpha}.$$

(ii) *If $\rho = 1/2$ then either H_n has infinitely many zeros or $\delta(\infty, f) < 1$.*

(iii) *If f is entire and $\rho < 1/2 + \delta_0$, then H_n has infinitely many zeros: here δ_0 is a small positive absolute constant.*

We remark that if f is an entire function for which H_n fails to be transcendental for some $n \geq 2$ then f satisfies a homogeneous linear difference equation with rational coefficients and the growth of such solutions was investigated in [12]. Moreover if f is a transcendental meromorphic function of order less than 1 then H_1 is always transcendental [2, 15]. The next result [2] applies to meromorphic functions with no assumption on the deficiency of the poles.

Theorem 1.3 ([2]) *Let f be a function transcendental and meromorphic in the plane of lower order $\lambda(f) < 1$. Let $c \in \mathbb{C} \setminus \{0\}$ be such that at most finitely many poles z_j, z_k of f satisfy $z_j - z_k = c$. Then $h(z) = f(z+c) - f(z)$ has infinitely many zeros.*

Clearly all but countably many $c \in \mathbb{C}$ satisfy the hypotheses of Theorem 1.3, but an example from [2] showed that Theorem 1.3 fails without the hypothesis on c , even for lower order 0. On the other hand for transcendental meromorphic functions of sufficiently small order, either the first difference or the first divided difference must have infinitely many zeros.

Theorem 1.4 ([14]) *Let f be a transcendental meromorphic function in the plane, of order less than $1/6$. Then at least one of Δf and $(\Delta f)/f$ has infinitely many zeros.*

The present paper was motivated by a question of Pat McCarthy at a recent conference in honour of Brian Twomey: can any of the above results be proved with the forward difference $f(z+1) - f(z)$ replaced by the so-called q -difference $f(qz) - f(z)$? For estimates for proximity functions involving the q -difference, as well as applications to q -difference equations, we refer the reader to [5]. We will prove the following theorem.

Theorem 1.5 *Let $q \in \mathbb{C}$ with $|q| > 1$. Let f be a transcendental meromorphic function in the plane with*

$$L(f) = \liminf_{r \rightarrow \infty} \frac{T(r, f)}{(\log r)^2} = 0, \tag{1.2}$$

and define F and G by

$$F(z) = f(qz) - f(z), \quad G(z) = \frac{F(z)}{f(z)}. \tag{1.3}$$

Then at least one of F and G has infinitely many zeros.

The following simple example shows that the hypothesis (1.2) is sharp in Theorem 1.5. Let $q \in \mathbb{C}$ with $|q| > 1$ and write

$$f(z) = \prod_{n=0}^{\infty} \left(1 - \frac{z}{q^n}\right)^{-1}, \quad f(qz) = \prod_{n=0}^{\infty} \left(1 - \frac{z}{q^{n+1}}\right)^{-1} = \frac{f(z)}{1 - qz}.$$

Hence G is a rational function and neither F nor G has infinitely many zeros, while $L(f)$ is positive but finite. On the other hand it seems plausible that under the hypotheses of Theorem 1.5 both F and G must have infinitely many zeros.

2 Preliminaries for Theorem 1.5

The proof of Theorem 1.5 will rest mainly on the following lemma from [13].

Lemma 2.1 *Let f be a transcendental meromorphic function in the plane and satisfy (1.2). Then there exist sequences (r_n) , (s_n) , (a_n) and (λ_n) with*

$$r_n, s_n \in (0, \infty), \quad \lim_{n \rightarrow \infty} r_n = \lim_{n \rightarrow \infty} s_n = \infty, \quad a_n \neq 0, \quad \lambda_n \in \mathbb{Z}, \quad (2.1)$$

such that, as $n \rightarrow \infty$,

$$f(z) = a_n z^{\lambda_n} (1 + \varepsilon(z)), \quad \text{where } \varepsilon(z) = \varepsilon_n(z) = o(1) \quad \text{for } r_n s_n^{-1} \leq |z| \leq r_n s_n. \quad (2.2)$$

Moreover it may be assumed that

$$\varepsilon'(z) = \varepsilon'_n(z) = o\left(\frac{1}{|z|}\right) \quad \text{for } r_n s_n^{-1} \leq |z| \leq r_n s_n. \quad (2.3)$$

The estimate (2.2) is proved in [13, Lemma 12]. To deduce (2.3) it is only necessary to replace s_n by $\sqrt{s_n}$ and apply Cauchy's estimate for derivatives. $\square$

Lemma 2.2 *Let $q \in \mathbb{C}$ with $|q| > 1$ and suppose that the sequences (r_n) , (s_n) , (a_n) and (λ_n) satisfy (2.1) with $\lambda_n \neq 0$. If $f_n(z)$ is analytic on $r_n s_n^{-1} \leq |z| \leq r_n s_n$ with $f_n(z) \sim a_n z^{\lambda_n}$ there, then there exists $d_n \in \mathbb{C} \setminus \{0\}$ such that*

$$f_n(qz) - f_n(z) \sim d_n z^{\lambda_n} \quad \text{for } |z| = r_n. \quad (2.4)$$

Proof. Let $|z| = r_n$. Then

$$f_n(qz) - f_n(z) = a_n q^{\lambda_n} z^{\lambda_n} (1 + o(1)) - a_n z^{\lambda_n} (1 + o(1)).$$

If $\lambda_n > 0$ this gives, since $|q| > 1$,

$$f_n(qz) - f_n(z) = a_n (q^{\lambda_n} - 1) z^{\lambda_n} + o(|a_n| q^{\lambda_n} |z|^{\lambda_n}) \sim a_n (q^{\lambda_n} - 1) z^{\lambda_n},$$

while for $\lambda_n < 0$ we have

$$f_n(qz) - f_n(z) = a_n (q^{\lambda_n} - 1) z^{\lambda_n} + o(|a_n| |z|^{\lambda_n}) \sim a_n (q^{\lambda_n} - 1) z^{\lambda_n}.$$

$\square$

Lemma 2.3 *Let f be a transcendental meromorphic function in the plane satisfying (1.2), and define G by (1.3). Then G is transcendental.*

Proof. This is standard. Assume that G is a rational function. Then there exists a rational function R such that $f(qz) = R(z)f(z)$. Take $s > 0$, so large that R has no zeros or poles in $|z| \geq s$. By (1.2) the function f must have infinitely many zeros or poles, and so there exists z_1 with $|z_1| > s$ such that z_1 is a zero or pole of f . But then so are $qz_1, q^2z_1, \dots$, and so

$$\liminf_{r \rightarrow \infty} \frac{N(r, f) + N(r, 1/f)}{(\log r)^2} > 0,$$

which contradicts (1.2). $\square$

Lemma 2.4 *Let f be a transcendental meromorphic function in the plane satisfying (1.2), and assume that f has either finitely many zeros in $\mathbb{C}$ or finitely many poles in $\mathbb{C}$. Define G by (1.3). Then G has infinitely many zeros in $\mathbb{C}$.*

Proof. Apply Lemma 2.1 to f . Since f has infinitely many zeros or poles in $\mathbb{C}$ by (1.2), it follows from (2.2) and the argument principle that

$$\lambda_n = n(r_n, 1/f) - n(r_n, f) \neq 0.$$

Since $|q| > 1$ this leads using Lemma 2.2 to (2.4), where $d_n \neq 0$. In particular we obtain

$$G(z) = \frac{F(z)}{f(z)} \sim \frac{d_n}{a_n} \quad \text{for } |z| = r_n,$$

and so the argument principle gives $n(r_n, 1/G) = n(r_n, G)$. But G must have infinitely many zeros or poles, by (1.2) and Lemma 2.3, and Lemma 2.4 is proved. $\square$

3 Proof of Theorem 1.5

Let f , F and G be as in the statement of Theorem 1.5, and assume that F and G both have finitely many zeros. Then by Lemma 2.4 we must have

$$\lim_{r \rightarrow \infty} n(r, 1/f) = \lim_{r \rightarrow \infty} n(r, f) = \infty. \quad (3.1)$$

Moreover, all but finitely many poles of f are poles of F of at least the same multiplicity, because otherwise they would be zeros of G . This gives positive constants A_0, A_1 such that

$$n(r, F) \geq n(r, f) - A_1 \quad \text{for } r \geq A_0. \quad (3.2)$$

Lemma 3.1 *There exist sequences (R_n) , (S_n) , (b_n) , (c_n) , and (μ_n) with*

$$R_n, S_n \in (0, \infty), \quad \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} S_n = \infty, \quad c_n \neq 0, \quad \mu_n \in \mathbb{Z} \setminus \{0\}, \quad (3.3)$$

such that

$$f(z) = b_n + c_n z^{\mu_n} (1 + \delta(z)) \quad \text{where } \delta(z) = \delta_n(z) = o(1) \quad \text{for } R_n S_n^{-1} \leq |z| \leq R_n S_n. \quad (3.4)$$

Proof. Apply Lemma 2.1 to f' . This gives sequences (r_n) , (s_n) , (a_n) and (λ_n) satisfying (2.1) such that

$$f'(z) = a_n z^{\lambda_n} (1 + \varepsilon(z)) \quad \text{where} \quad \varepsilon(z) = \varepsilon_n(z) = o(1) \quad \text{for} \quad r_n s_n^{-1} \leq |z| \leq r_n s_n. \quad (3.5)$$

Moreover it may be assumed that (2.3) also holds. Then $\lambda_n \neq -1$ in (3.5), since otherwise the residue theorem gives

$$0 = \int_{|z|=r_n} f'(z) dz \sim 2\pi i a_n,$$

a contradiction. Taking z as in (3.5) and integrating by parts from r_n to z we obtain, for some $b_n \in \mathbb{C}$,

$$f(z) = b_n + c_n z^{\mu_n} (1 + \varepsilon(z)) - R(z)$$

where, using (2.3),

$$\mu_n = \lambda_n + 1 \neq 0, \quad c_n = \frac{a_n}{\mu_n} \quad \text{and} \quad R(z) = \int_{r_n}^z c_n t^{\mu_n} o(|t|^{-1}) dt.$$

Here the path of integration may be taken to be an arc of the circle $|t| = r_n$ joining r_n to $z^* = z r_n / |z|$ followed by the radial segment from z^* to z .

If $\mu_n > 0$ then we take z with $r_n \leq |z| \leq r_n s_n$ and we obtain $|R(z)| = o(|c_n| |z|^{\mu_n})$ as required. Finally if $\mu_n < 0$ then we take z with $r_n s_n^{-1} \leq |z| \leq r_n$ and this time

$$|R(z)| \leq o(|c_n| r_n^{\mu_n}) + \int_{|z|}^{r_n} o(|c_n| s^{\mu_n-1}) ds = o(|c_n| |z|^{\mu_n}).$$

This proves Lemma 3.1. □

Lemma 3.2 *With the same notation as in Lemma 3.1, assume without loss of generality that $\lim_{n \rightarrow \infty} b_n = b \in \mathbb{C} \cup \{\infty\}$. Then the equation $f(z) = b$ has finitely many solutions in $\mathbb{C}$.*

Proof. Set $g_n = f - b_n$. Then Lemma 2.2 and (3.4) give $d_n \neq 0$ such that

$$F(z) = f(qz) - f(z) = g_n(qz) - g_n(z) \sim d_n z^{\mu_n} \quad \text{for} \quad |z| = R_n.$$

Hence we obtain, using (3.4) again,

$$\begin{aligned} \mu_n &= n(R_n, 1/F) - n(R_n, F) \\ &= n(R_n, 1/g_n) - n(R_n, g_n) \\ &= n(R_n, 1/g_n) - n(R_n, f). \end{aligned}$$

Using (3.2) and the assumption that F has finitely many zeros this gives $A_2 > 0$, independent of n , such that

$$n(R_n, 1/g_n) \leq A_2. \quad (3.6)$$

Now suppose that the equation $f(z) = b$ has infinitely many solutions in $\mathbb{C}$. Then there exists $T > 0$ such that the equation $f(z) = b$ has at least $A_2 + 1$ distinct solutions in $|z| < T$, and

for large n so has the equation $f(z) = b_n$, by Rouché's theorem. But this implies that, for n sufficiently large,

$$n(R_n, 1/g_n) \geq n(T, 1/g_n) \geq A_2 + 1,$$

contradicting (3.6). $\square$

It follows from Lemma 3.2 and (3.1) that $b \in \mathbb{C} \setminus \{0\}$. Applying Lemma 2.4 to $h = f - b$ then shows that the function

$$H(z) = \frac{F(z)}{f(z) - b} = \frac{h(qz) - h(z)}{h(z)}$$

has infinitely many zeros. If these are poles of f then they are zeros of

$$G = \frac{F}{f} = H \cdot \frac{f - b}{f},$$

while if they are not poles of f then they are zeros of F . This completes the proof of Theorem 1.5. $\square$

Corollary 3.1 *Let f be a transcendental meromorphic function in the plane satisfying (1.2), and let $a, b \in \mathbb{C}$ with $|a| \neq 0, 1$. Then at least one of $H(z) = f(az + b) - f(z)$ and $H(z)/f(z)$ has infinitely many zeros.*

Proof. We begin by showing that in Theorem 1.5 the hypothesis $|q| > 1$ may be replaced by $0 < |q| < 1$. To see this, let f be as in the hypotheses of Theorem 1.5, let q be a complex number with $0 < |q| < 1$ and set $p = 1/q$ and $w = pz$. If $f(pz) - f(z) = f(w) - f(qw)$ has infinitely many zeros then obviously so has $f(qw) - f(w)$, so assume that this is not the case. Then, by Theorem 1.5,

$$\frac{f(pz) - f(z)}{f(z)}$$

must have infinitely many zeros z_k which are poles of $f(z)$ of multiplicity m_k but poles of $f(pz) - f(z)$ of multiplicity less than m_k (possibly zero). Hence each z_k must be a pole of $f(pz)$ of multiplicity m_k and so a zero of

$$\frac{f(pz) - f(z)}{f(pz)} = \frac{f(w) - f(qw)}{f(w)}.$$

This proves our assertion. To complete the proof of Corollary 3.1 choose $d \in \mathbb{C}$ with $ad + b = d$ and set $z = t + d$ and $f_1(t) = f(t + d)$. Then

$$H(z) = f(a(t + d) + b) - f(t + d) = f(at + d) - f(t + d) = f_1(at) - f_1(t)$$

and $H(z)/f(z) = (f_1(at) - f_1(t))/f_1(t)$. Furthermore, (1.2) gives a sequence $r_n \rightarrow \infty$ with

$$n(4r_n, f) + n(4r_n, 1/f) + n(4r_n, 1/(f - 1)) = o(\log r_n).$$

Hence the second fundamental theorem gives $s_n \in [r_n, 2r_n]$ such that

$$\begin{aligned}
(1 - o(1))T(s_n, f_1) &\leq N(s_n, f_1) + N(s_n, 1/f_1) + N(s_n, 1/(f_1 - 1)) \\
&\leq (n(2r_n, f_1) + n(2r_n, 1/f_1) + n(2r_n, 1/(f_1 - 1))) \log 2r_n + O(1) \\
&\leq (n(4r_n, f) + n(4r_n, 1/f) + n(4r_n, 1/(f - 1))) \log 2r_n + O(1) \\
&= o(\log s_n)^2.
\end{aligned}$$

This shows that (1.2) holds with f replaced by f_1 . □

We close by remarking that if q is a root of unity then it is easy to choose f such that both $F(z) = f(qz) - f(z)$ and $F(z)/f(z)$ have no zeros, while if $|q| = 1$ but q is not a root of unity then the situation is unclear: in particular the proof of Lemma 2.2 breaks down because if $|\lambda_n|$ is large then $q^{\lambda_n} - 1$ may be small.

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