# Complex oscillation and removable sets

J.K. Langley

**Abstract**

Let $A$ be a transcendental entire function of finite order, and let $E$ be the product of linearly independent solutions of $w'' + A(z)w = 0$. We prove the existence of sequences of annuli $\Omega_m$ such that if $E$ has relatively few zeros in the union of the $\Omega_m$, then $E$ has relatively few zeros in the whole plane.

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## 1 Introduction

Let $A$ be an entire function, and let $f_1, f_2$ be linearly independent solutions of the equation

$$w'' + A(z)w = 0, \tag{1}$$

normalized so that the Wronskian $W = W(f_1, f_2) = f_1 f_2' - f_1' f_2$ satisfies $W = 1$. The Bank-Laine product function $E = f_1 f_2$ satisfies $E'(z) = \pm 1$ at every zero $z$ of $E$, as well as the relation

$$4A = (E'/E)^2 - 2E''/E - 1/E^2. \tag{2}$$

Conversely, if $E$ is any entire function with the property that $E'(z) = \pm 1$ at every zero $z$ of $E$, then [2] the function $A$ defined by (2) is entire, and $E$ is the product of linearly independent normalized solutions of (1).

Extensive work in recent years has concerned the exponent of convergence $\lambda(f_j)$ of the zeros of solutions $f_j$, in connection with the order of growth $\rho(A)$ of the coefficient $A$, these defined by

$$\lambda(f_j) = \limsup_{r \to \infty} \frac{\log^+ N(r, 1/f_j)}{\log r}, \quad \rho(A) = \limsup_{r \to \infty} \frac{\log^+ T(r, A)}{\log r}.$$

Note that

$$\rho(E) \geq \lambda(E) = \max\{\lambda(f_1), \lambda(f_2)\}.$$

It has been conjectured that the condition

$$A \quad \text{transcendental}, \quad \rho(A) < \infty, \quad \lambda(E) < \infty \tag{3}$$

implies that $\rho(A)$ is a positive integer, and it has been shown [14, 15] that (3) implies that $\rho(A) > 1/2$ and that $E$ has finite order [1]. Further results may be found in [3, 4, 10, 13] and elsewhere.

The present paper is concerned with a problem first considered in [11], that of the existence of sets $D$ which are removable in the following sense: if $A$ has finite order and the zeros of $E$ outside $D$ have finite exponent of convergence then $\rho(E)$ is finite, and hence so is $\lambda(E)$.

**Theorem A [11].** *Let $\eta, K$ and $S$ be constants with $0 < \eta < \pi$ and $1 < K < S$. Let $R_m$ be a positive sequence tending to infinity with $R_{m+1} > SR_m$ for each positive integer $m$, and let $\phi_m$ be a real sequence. Let $D$ be the union of the*

$$D_m = \{z = re^{i\theta} : R_m < r < KR_m, \phi_m - \eta < \theta < \phi_m + \eta\}.$$

*Suppose that $A$ is a transcendental entire function of finite order and that $E = f_1 f_2$ is the product of linearly independent normalized solutions $f_j$ of* (1)*, and suppose finally that the zeros of $E$ in the complement of $D$ have finite exponent of convergence. Then $E$ has finite order.*

Thus if $E$ has relatively few zeros outside $D$ then $E$ has relatively few in the whole plane. A drawback of Theorem A is that the complement of the removable set $D$ needs to be connected in order for the proof in [11], which is based in part on the methods of [18], to work. A further result [11, Theorem 3] eliminated this restriction, but subject to the strong additional hypothesis that there exists at least one ray $\arg z = \theta$ along which $A(z)$ has polynomial growth, i.e. $\log^+ |A(re^{i\theta})| = O(\log r)$. We show here that the logarithmic rectangles occurring in Theorem A may be replaced by annuli, with no hypothesis on the coefficient function $A$ other than that $A$ is transcendental of finite order, the resulting removable set having disconnected complement. Our approach is more direct than that of [11], using in part a simplified version of the method of [12].

**Theorem 1.** *Suppose that $K$ and $M$ are positive constants with $K > 1$, that $A$ is a transcendental entire function of finite order and that $E = f_1 f_2$ is the product of linearly independent normalized solutions of* (1)*. Suppose that there exists a positive sequence $r_m$ tending to infinity such that for each large positive integer $m$ the number of zeros of $E$ in the annulus*

$$\Omega(r_m, K) = \{z : K^{-1} r_m < |z| < K r_m\}$$

*is at most $(r_m)^M$. Then*

$$\log T(K^{-1} r_m, E) = O(\log r_m), \quad m \to \infty. \tag{4}$$

**Theorem 2.** *Suppose that $K, M, A, E, r_m$ are as in the hypotheses of Theorem* 1*, and assume in addition that*

$$\limsup_{m \to \infty} \frac{\log r_{m+1}}{\log r_m} < \infty. \tag{5}$$

*Then $E$ has finite order.*

Thus with the hypotheses of Theorem 2, the complement of the union of the $\Omega(r_m, K)$ is a removable set in the sense described above. The hypothesis in Theorems 1 and 2 that $A$ has finite order is not redundant. Let $H$ be an entire function, having $\exp(2^m)$ simple zeros on the circle $|z| = 2^m$, for each positive integer $m$, and no other zeros. As in [16] choose, using Mittag-Leffler interpolation, an entire function $g$ such that $E = He^g$ satisfies $E'(z) = \pm 1$ at each of these zeros. Then $E$ is the product of linearly independent normalized solutions of (1), and $\lambda(E) = \infty$, although $E$ has no zeros in the annuli $2^m < |z| < 2^{m+1}$.

## 2 Lemmas needed for the proof of Theorem 1

**Lemma 1.** *Suppose that $K, M, A, E, r_m$ are as in the hypotheses of Theorem* 1*. Then there exist positive constants $M_1, M_2$ with the following properties.*

*If $m$ is a sufficiently large positive integer there exists $v_m$ in $\Omega(r_m, K^{1/4})$ such that*

$$|\log |E(v_m)|| \le (r_m)^{M_1}, \tag{6}$$

*and such that $E$ has no zeros in the disc $B(v_m, (r_m)^{-M_2})$.*

*Proof.* We use $c$ to denote a positive constant not depending on $m$, not necessarily the same at each occurrence. We note first that, by [12, p.508], or using Herold's comparison theorem [7], the normalized solutions $f_j$ of (1) satisfy, for large $r$,

$$|f_j(z)| + |f_j'(z)| \leq \exp(3rM(r,A)^{1/2}), \quad |z| \leq r. \tag{7}$$

We may choose $s$ with $K^{-1/8}r_m < s < K^{1/8}r_m$, such that $s$ is normal for $A$ with respect to the Wiman-Valiron theory [6, 17]. This means that if $|z_0| = s$ and $|A(z_0)| = M(s,A)$, then we have

$$A(z) = (z/z_0)^N A(z_0)(1 + o(1)), \tag{8}$$

and

$$A'(z)/A(z) = (1 + o(1))N/z, \quad A''(z)/A(z) = (1 + o(1))N^2/z^2 \tag{9}$$

for $z$ in $D(z_0, 2)$, in which

$$D(z_0, L) = \{z = z_0 e^\tau : |Re(\tau)| \leq LN^{-5/8}, |Im(\tau)| \leq LN^{-5/8}\}. \tag{10}$$

Here $N = \nu(s,A)$ is the central index of $A$ and, provided $s$ lies outside a set of finite logarithmic measure, may be assumed to satisfy

$$N \leq (\log M(s,A))^{5/4}. \tag{11}$$

Define

$$z_1 = z_0 \exp(-N^{-5/8}), \quad Z = 2A(z_1)^{1/2} z_1/(N+2) + \int_{z_1}^{z} A(t)^{1/2} dt. \tag{12}$$

We may write (8) in the form

$$A(z) = (z/z_1)^N A(z_1)(1 + \mu(z))^2, \quad \mu(z) = o(1),$$

for $z$ in $D(z_0, 2)$, so that $\mu'(z) = o(N^{5/8}s^{-1})$ for $z$ in $D(z_0, 1)$. Thus for $z$ in $D(z_0, 1)$, integration by parts from $z_1$ to $z$ along part of the ray $\arg t = \arg z_1$ and part of the circle $|t| = |z|$, this path having length $O(sN^{-5/8})$, gives

$$\int_{z_1}^{z} t^{N/2}\mu(t)dt = o((N+2)^{-1}|z|^{(N+2)/2}) - \int_{z_1}^{z} 2(N+2)^{-1} t^{(N+2)/2} o(N^{5/8}s^{-1})dt =$$

$$= o((N+2)^{-1}|z|^{(N+2)/2}).$$

Hence

$$Z = (1 + o(1))A(z_1)^{1/2} 2z^{(N+2)/2}(z_1)^{-N/2}(N+2)^{-1} = (1 + o(1))A(z)^{1/2} 2z(N+2)^{-1}, \tag{13}$$

for $z$ in $D(z_0, 1)$. Further,

$$Z(z)/Z(z_0) = (1 + o(1))(z/z_0)^{(N+2)/2}$$

for $z$ in $D(z_0, 1)$ and, since $Z$ is locally univalent, by (12), the function $Z$ has, in $D(z_0, 1/2)$, at least one simple island $H_0$ mapped univalently onto the closed region $H_1$ given by

$$|\log|Z/Z_0|| \leq N^{1/3}, \quad |\arg Z| \leq \pi/4, \quad Z_0 = |A(z_0)^{1/2} 2z_0(N+2)^{-1}| = (1 + o(1))|Z(z_0)|. \tag{14}$$

By (11), $Z_0 \exp(-N^{1/3})$ is large, when $s$ is large enough.

As the next step in the proof of Lemma 1 we apply a local analogue of Hille's asymptotic method [8, 9] developed in [12]. We write

$$W(Z) = A(z)^{1/4} w(z), \tag{15}$$

for $z$ in $H_0$ and $Z$ in $H_1$, in which $w$ is a solution of (1). The equation (1) transforms to

$$\frac{d^2W}{dZ^2} + (1 - F_0(Z))W = 0, \quad F_0(Z) = A''(z)/4A(z)^2 - 5A'(z)^2/16A(z)^3. \tag{16}$$

By (9), we have $|F_0(Z)| \leq 3|Z|^{-2}$ in $H_1$. By [12, Lemma 1] there exist solutions $U_1(Z), U_2(Z)$ of (16) satisfying

$$U_j(Z) = (1+o(1))\exp((-1)^j iZ), \quad U_j'(Z) = (1+o(1))(-1)^j i \exp((-1)^j iZ), \quad W(U_1, U_2) = 2i+o(1) \tag{17}$$

in $H_1$. We write, in $H_0$, for $q = 1, 2$,

$$f_q(z) = C_q u_1(z) + D_q u_2(z), \quad u_j(z) = A(z)^{-1/4} U_j(Z), \tag{18}$$

with $C_1, D_1, C_2, D_2$ constants. Choose $z^*$ in $H_0$ so that $Z^* = Z(z^*)$ satisfies

$$|Z^*| \leq (1/2)Z_0, \quad |U_2(Z^*)| \leq 2, \quad |U_2'(Z^*)| \leq 2.$$

Then using (9), (11), (14), (17) and (18) we have

$$|z^*| \leq s, \quad u_2(z^*) = o(1), \quad |u_2'(z^*)/u_2(z^*)| \leq N + 2M(s, A)^{1/2} \leq 3M(s, A)^{1/2}. \tag{19}$$

Further, (17) and standard properties of Wronskians give

$$W(u_1, u_2) = 2i + o(1)$$

in $H_0$. Thus the equation

$$C_1 = W(f_1, u_2)/W(u_1, u_2)$$

and (7) and (19) give

$$|C_1| \leq cM(s, A)^{1/2} \exp(3sM(s, A)^{1/2}) \leq M = \exp(4NZ_0), \tag{20}$$

using (14), and the same estimate holds for $C_2, D_1, D_2$.

We require further estimates for the coefficients $C_1, C_2, D_1, D_2$ and for convenience we state and prove these as Lemma 2, following which the proof of Lemma 1 will be completed.

**Lemma 2.** *In each pair $\{C_1, D_1\}, \{C_2, D_2\}$, at least one term has modulus at most $M^{-2}$.*

*Proof.* Suppose that $C_1$ and $D_1$ each have modulus at least $M^{-2}$. Set $F_1(Z) = A(z)^{1/4} f_1(z)$, for $z$ in $H_0$, and $Z$ in $H_1$. Then we may write, using (18) and (20),

$$F_1(Z) = -C_1 U_1(Z)(e^{2iY} - 1), \quad Y = Z + S + o(1), \quad |S| \leq 32NZ_0. \tag{21}$$

It follows from (14), (20) and (21) that the image of $H_1$ under $Y$ covers the region

$$(1/4)N^{1/3} \leq \log|Y/Z_0| \leq (1/2)N^{1/3}, \quad |\arg Y| \leq \pi/8,$$

so that the number of zeros of $f_1$ in $H_0$ is at least $\exp(cN^{1/3}Z_0)$. Since every zero of $f_1$ is a zero of $E$ and, since $H_0$ is contained in $\Omega(r_m, K)$, this contradicts the hypotheses of Theorem 1. Lemma 2 is proved.

We return to the proof of Lemma 1. Since $F_1F_2$ has at most $(r_m)^M$ zeros in $H_1$, we may choose $Z_1$ in $H_1$ with

$$(1/4)e^{N^{1/3}}Z_0 \leq |Z_1| \leq (3/4)e^{N^{1/3}}Z_0$$

and with

$$|U_1(Z_1)| = 1 + o(1), \quad |U_2(Z_1)| = 1 + o(1), \tag{22}$$

and such that $F_1F_2$ has no zeros in the region

$$\{Z : |\log(Z/Z_1)| \leq N^{1/3}(r_m)^{-2M}\}.$$

Let $v_m$ be the pre-image of $Z_1$ in $H_0$. Since, by (12) and (13),

$$Z^{-1}dZ/dz = (1 + o(1))(N + 2)/2z$$

on $D(z_0, 1)$, it follows that $E = f_1f_2$ has no zeros $\zeta$ with

$$|\log(\zeta/v_m)| \leq \lambda, \quad \lambda = N^{1/3}(r_m)^{-2M}(N + 2)^{-1} \geq (r_m)^{-c}.$$

In the last inequality we have used (11).

It remains only to estimate $E(v_m)$. Since

$$1 = W(f_1, f_2) = (C_1D_2 - C_2D_1)W(u_1, u_2) = (C_1D_2 - C_2D_1)(2i + o(1))$$

in $H_0$, it follows from (20) and Lemma 2 that either $C_1$ and $D_2$ each have modulus at most $M^{-2}$, or $C_2$ and $D_1$ each have modulus at most $M^{-2}$. We assume without loss of generality that the latter is the case. We therefore have

$$C_1C_2 = o(1), D_1C_2 = o(1), \quad D_1D_2 = o(1), \quad C_1D_2 = (1/2i)(1 + o(1)).$$

Further,

$$E = (C_1u_1 + D_1u_2)(C_2u_1 + D_2u_2) = (C_1D_2 + D_1C_2)u_1u_2 + C_1C_2u_1^2 + D_1D_2u_2^2$$

so that using (22) we have

$$E(v_m) = (1 + o(1))(1/2i)u_1(v_m)u_2(v_m) = (1 + o(1))(1/2i)A(v_m)^{-1/2}$$

and Lemma 1 is proved.

## 3 Proof of Theorems 1 and 2

Suppose that $K, M, A, E, r_m$ are as in the statement of Theorem 1, and that $v_m$ and $M_1, M_2$ are as in Lemma 1. Let $\phi(z)$ map the unit disc $\Delta = B(0, 1)$ conformally onto the logarithmic rectangle $U = \{w : |\log|w|| < (1/4)\log K, |\arg w| < \pi\}$, with $\phi(0) = 1$, and let

$$h(z) = h_m(z) = v_m\phi(z)^2. \tag{23}$$

Defining

$$B(z) = A(h(z)), \quad F(z) = E(h(z)), \tag{24}$$

we have, by (2),

$$(h')^2F^{-2} = (F'/F)^2 - 2F''/F + 2(h''/h')(F'/F) - 4(h')^2B \tag{25}$$

for $z$ in $\Delta$. Since $A$ has finite order we have, by (7),

$$\log^+ |B(z)| + \log^+ \log^+ |F(z)| \leq (r_m)^d \tag{26}$$

for $z$ in $\Delta$, using $d$ to denote a positive constant not depending on $m$, not necessarily the same at each occurrence. Using (6) and (26), we have

$$\log^+ T(r, 1/F) \leq \log^+ T(r, F) + (r_m)^d \leq (r_m)^d, \quad 0 < r < 1. \tag{27}$$

Since $E$ has no zeros in the disc $B(v_m, (r_m)^{-M_2})$, and at most $(r_m)^M$ zeros in the annulus $\Omega(r_m, K)$, we have

$$N(r, 1/F) \leq (r_m)^d, \quad 0 < r < 1. \tag{28}$$

Choose $s_1$ in $(0,1)$ such that the image of the disc $B(0, s_1)$ under $\phi$ contains the arc $|u| = 1, |\arg u| \leq \pi/2$, and define $s_j$ for $2 \leq j \leq 4$ by $s_j = (1 + s_{j-1})/2$. Again since $F$ has at most $2(r_m)^M$ zeros in $\Delta$, we may choose $r$ with $s_2 \leq r \leq s_3$ such that $F$ has no zeros $\zeta$ with $||\zeta| - r| < {r_m}^{-2M}$. Choose $R$ with $s_4 < R < 1$ such that $F$ has no zeros on $|\zeta| = R$, and apply the differentiated Poisson-Jensen formula [5, p.22] to $F$ in $B(0, R)$ in order to estimate $F'(z)/F(z)$ and $(F'/F)'(z)$, with $|z| = r$. This gives

$$|F'(z)/F(z)| + |(F'/F)'(z)| \leq (r_m)^d + (r_m)^d (m(R, F) + m(R, 1/F)), \quad |z| = r,$$

and so, using (27),

$$m(r, F'/F) + m(r, F''/F) \leq (r_m)^d.$$

Using (23), (24), (25), (26), and (28) we now have

$$T(r, 1/F) \leq (r_m)^d$$

and hence, using (6) again,

$$T(r, F) \leq (r_m)^d$$

from which it follows that

$$\log |F(z)| \leq (r_m)^d, \quad |z| \leq s_1,$$

and, by the choice of $s_1$,

$$\log |E(w)| \leq (r_m)^d, \quad |w| = |v_m|.$$

Since the sequence $|v_m|$ satisfies $K^{-1/4} r_m < |v_m| < K^{1/4} r_m$, we obtain (4) and Theorem 1. If, in addition, we have (5), it follows at once that $E$ has finite order, and Theorem 2 is proved.

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School of Mathematical Sciences, University of Nottingham, NG7 2RD.