On multiple points of meromorphic functions

Jim Langley and Dan Shea

Abstract

We establish lower bounds for the number of zeros of the logarithmic derivative of a meromorphic function of small growth with few poles.

1 Introduction

Our starting point is the following theorem, a combination of results from [5] and [9], in which the order of growth of a function f meromorphic in the plane is defined by $\rho(f) = \limsup_{r\to\infty} \log T(r,f)/\log r$, the corresponding lower order having lim inf in place of lim sup.

Theorem A *Suppose that f is a function transcendental and meromorphic in the plane, of order ρ.*
(i) If $\rho < 1/2$ then f'/f has infinitely many zeros.
(ii) If $\rho < 1$ and f is entire then f'/f has infinitely many zeros.
(iii) If $\rho < 1$ then f' has infinitely many zeros.

A unified proof of these results was given in the paper [3]. There is a long tradition in value distribution theory of results in which integer or half-integer orders are exceptional, going back to the classical factorization theorem of Hadamard. However, in addition to the examples $\tan^2 \sqrt{z}$ and e^z, all three parts of the above theorem are fully sharp in the sense that it is shown in [5] that there exist examples with f'/f (or f') zero-free, for each order of growth at least the bound given.

Theorem A may be interpreted as saying the following. A transcendental meromorphic function f of order less than 1 cannot be such that all but finitely many multiple points lie over one value, so that f' has zeros and, if f has no poles, f'/f must have zeros. Further, if f has order less than $1/2$, then it is not possible for all but finitely many multiple points to lie over two values. Indeed, if the transcendental function f is entire with $\log M(r,f) = o(r^{1/2})$ or meromorphic with $T(r,f) = o(\log^2 r)$ then f must have infinitely many critical values [3, 16].

We remark further that the paper [5] was partly inspired by a question of L.A. Rubel [4], and both [5] and [9] contain results dealing with critical, or equilibrium, points of delta-subharmonic functions of the form

$$u(z) = \sum_{k=1}^{\infty} a_k \log|1 - z/z_k|, \quad \sum_{k=1}^{\infty} |a_k/z_k| < \infty, \tag{1}$$

these critical points being zeros of the meromorphic function

$$G(z) = \sum_{k=1}^{\infty} \frac{a_k}{z - z_k}. \tag{2}$$

Now, if f is transcendental and meromorphic in the plane of order less than 1 then by [8, 19] the counting function of the multiple points of f cannot be $o(T(r, f))$. With this in mind, the present paper is concerned with attempting to improve the conclusions of Theorem A to give a lower estimate for the number of zeros, some results in this direction already having appeared in [9]. It was shown there that if f is transcendental entire of order less than 1, having zeros of bounded multiplicity, then $\delta(0, f'/f) < 1$. In addition, the following theorem was proved in [9].

Theorem B *Suppose that f is an entire function such that f'/f is transcendental of lower order $\mu \leq 1/2$. If $\mu < 1/2$ then the Nevanlinna deficiency $\delta(0, f'/f)$ of the zeros of f'/f satisfies $\delta(0, f'/f) \leq 1 - \cos \pi\mu$. If $\mu = 1/2$ then $\delta(0, f'/f) < 1$.*

There is a conjecture of Fuchs that if f is a transcendental entire function of lower order less than 1/2, then $\delta(0, f'/f) = 0$, and Theorem B proves this for $\mu = 0$. The proof of Theorem B for $\mu < 1/2$ is based on a theorem of Gol'dberg [11]. Assuming that $\delta(0, f'/f) > 1 - \cos \pi\mu$, there exists a sequence $r_n \to \infty$ such that the minimum modulus of f/f' on $|z| = r_n$ tends to infinity rapidly as $n \to \infty$: this contradicts the fact that, for an entire function f, the logarithmic derivative must be not too small at some point on each such circle. Using the following result of Gol'dberg and Sokolovskaya [12] and an argument based on the derivative of the Nevanlinna characteristic, we will prove, for $\mu < 1/2$, an analogue of Theorem B for meromorphic functions with deficient poles.

Theorem C *Suppose that f is transcendental and meromorphic in the plane, of lower order $\mu < \alpha < 1$, and define*

$$G_\alpha(f) = \{r > 1 : \log L(r, f) > \gamma(\cos \pi\alpha N(r, 1/f) - N(r, f))\}, \tag{3}$$

in which $L(r, f) = \min\{|f(z)| : |z| = r\}$ and $\gamma = \pi\alpha / \sin \pi\alpha$. Then $G_\alpha(f)$ has upper logarithmic density at least $1 - \mu/\alpha$.

We refer the reader to [1, 11, 12, 15] for related minimum modulus results of $\cos \pi\rho$ type, and we remark that an alternative proof of Theorem C appeared in an earlier version of this paper, written before we learned of [12]. Our first result on the zeros of logarithmic derivatives of meromorphic functions is the following.

Theorem 1 *Let f be a transcendental meromorphic function such that f'/f is transcendental of lower order $\mu < \alpha < 1/2$. Then*

$$\delta(\infty, f) \leq \mu/\alpha \qquad \textit{or} \qquad \delta(0, f'/f) \leq 1 - \cos \pi\alpha.$$

The example following Theorem 2 shows that $\delta(0, f'/f) = 1$ is possible when f is meromorphic without deficient poles. We turn our attention now to the case where f'/f has lower order at least $1/2$, and first recall one approach to proving Theorem A, part (ii). Suppose that f is transcendental entire, of order less than 1, and that f'/f has no zeros. Then f/f' is transcendental entire, and there exists an unbounded domain U_1 on which f'/f is very small, so that "in principle" one can deduce that f tends to a finite asymptotic value in U_1. On the other hand, f must be large in some unbounded region U_2 and, because of the slow growth of f (and f'/f), both U_1 and U_2 must have, on average, angular measure greater than π, this being of course impossible. It was proved by Hayman in [14] (Corollary 4) that if g is meromorphic in the plane with lower order at least $1/2$, while the poles of g have order less than $1/2$, then ∞ is an asymptotic value of g, and this suggests an approach similar to that above may be applied, when f has few poles. There are a number of difficulties in doing this, since we need information on the rate of growth of f/f' and the length of the path, which Hayman's result does not give in all cases. The key fact that we use from [14] is the following: if g has sufficiently few poles and is bounded on a path tending to infinity, then $\log^+ |g(z)|$ has a non-constant subharmonic minorant. We combine this with a modification of the length-area technique due to Weitsman [9, 21]. Before stating our theorem, we remark that a result of Eremenko [7] also gives sufficient conditions for ∞ to be an asymptotic value, but does not seem so suitable for application to our present problem.

Theorem 2 *Suppose that d is an odd positive integer and that f is transcendental and meromorphic in the plane, with $T(r, f) = O(r^d)$ as $r \to \infty$, and*

$$\liminf_{r\to\infty} T(r, f)/r = 0.$$

Then either

$$\liminf_{r\to\infty} \left(T(r, f) - (1/2)r^{1/2} \int_r^\infty \frac{N(t, f)}{t^{3/2}} dt \right) < \infty, \tag{4}$$

or

$$\liminf_{r\to\infty} \left(T(r, f'/f) - (1/2)r^{1/2} \int_r^\infty \frac{N(t, f/f')}{t^{3/2}}\, dt - \frac{5+d}{2} \log r \right) < \infty. \tag{5}$$

Corollary *Suppose that f is transcendental meromorphic in the plane, of finite order, with lower order $\mu < 1$. If f'/f has lower order $\lambda \geq 1/2$, then $N(r,f) + N(r,f/f')$ has order at least $1/2$.*

It is natural to ask whether some analogous theorems hold, related to part (iii) of Theorem A, giving lower bounds for the number of zeros of f' in terms of the growth of f or f'/f, but the following fairly well known construction shows that no such results are possible. Take a meromorphic function g, of finite order, with no zeros and with poles whose multiplicities tend to infinity rapidly, so that $\overline{N}(r,g)$ is small compared to $T(r,g)$, and so is $T(r,g'/g)$, since

$$T(r,g'/g) \leq \overline{N}(r,g) + O(\log r).$$

We choose a such that $\overline{N}(r,1/(g+a)) = (1+o(1))T(r,g)$, and such that $-a$ is not a critical value of g, and we set $f = g + a$. We then have

$$N(r,1/f') = N(r,g/g') = o(T(r,f))$$

and

$$T(r,f'/f) \geq \overline{N}(r,1/f) = (1+o(1))T(r,g) = (1+o(1))T(r,f)$$

so that $N(r,1/f')$ is small compared to $T(r,f)$ and $T(r,f'/f)$. Indeed, we can ensure that $T(r,f)$ and $T(r,f'/f)$ both have positive lower order, while $N(r,1/f')$ has order 0. The following theorem essentially pinpoints these examples, showing that if f' has few enough zeros, then $f'/(f-a)$ must have comparatively small growth, for some finite a.

Theorem 3 *Suppose that d is an odd positive integer and that f is transcendental and meromorphic in the plane with $T(r,f) = O(r^d)$ as $r \to \infty$ and*

$$\liminf_{r\to\infty} T(r,f)/r = 0.$$

Then either

$$\liminf_{r\to\infty} \left(T(r,f') - (1/2)r^{1/2} \int_r^\infty \frac{N(t,1/f')}{t^{3/2}} dt - \frac{5+d}{2} \log r \right) < \infty \tag{6}$$

or, for some complex number a,

$$\liminf_{r\to\infty} \left(T(r,f'/(f-a)) - (1/2)r^{1/2} \int_r^\infty \frac{N(t,(f-a)/f')}{t^{3/2}} dt - \frac{5+d}{2} \log r \right) < \infty. \tag{7}$$

Of course, $N(t,(f-a)/f') \leq N(t,1/f')$ in (7).

Corollary *Suppose that f is transcendental and meromorphic in the plane, of finite order, with lower order $\mu < 1$. If $\overline{N}(r,f)$ has lower order λ, then $N(r,1/f')$ has order at least* $\min\{1/2,\lambda\}$.

The corollaries to Theorems 2 and 3 may be interpreted for functions of finite order with lower order less than 1 as follows. If f has few poles, counting multiplicity, then f'/f has a lot of zeros while, if f has poles at sufficiently many points, then f' must have a lot of zeros. The proof of Theorem 3 is rather more difficult than that of Theorem 2, and so we present the former in detail in Section 4, outlining the latter in Section 5.

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2 Proof of Theorem 1

We begin by observing that Theorem C leads at once to the following: if g is transcendental and meromorphic in the plane with lower order μ satisfying $\mu < \alpha < 1/2$, if

$$\delta(\infty,g) > 1-\cos\pi\alpha, \tag{8}$$

and if the set F is defined by

$$F=\{r>1:\log L(r,g) > \gamma(\cos\pi\alpha+\delta(\infty,g)-1)T(r,g)\}, \tag{9}$$

in which $L(r,g)$ and γ are as in Theorem C, then F has upper logarithmic density at least $1-\mu/\alpha$. We deduce this in the following standard way, choosing c such that $N(r,1/(g-c)) = (1+o(1))T(r,g)$. We set $g_1(z) = (g(z)-c)c_1z^{c_2}$ for suitable c_1, c_2 so that $g_1(0)=1$, and we have $T(r,g_1) = T(r,g)+O(\log r)$ and $\delta(\infty,g_1) = \delta(\infty,g)$. We note that, using (8), the function

$$s(t) = \frac{\pi t}{\sin\pi t}\left(\cos\pi t+\delta(\infty,g)-1\right)$$

is positive and decreasing for t close to α. We choose α' slightly smaller than α, with $\gamma' = \pi\alpha'/\sin\pi\alpha'$, and we note that

$$\gamma'\left(\cos\pi\alpha' N(r,1/g_1) - N(r,g_1)\right) \geq$$

$$\gamma'\left(\cos\pi\alpha'+\delta(\infty,g)-1-o(1)\right)T(r,g) \geq (s(\alpha)+\varepsilon)T(r,g)$$

for some positive ε and for all large r. We thus have, by Theorem C,

$$\log L(r,g_1) \geq (s(\alpha)+\varepsilon)T(r,g)$$

on a set of $r \geq 1$ of upper logarithmic density at least $1 - \mu/\alpha'$, and for these r we have

$$\log L(r, g) \geq \log L(r, g_1) - O(\log r).$$

Since α' may be chosen arbitrarily close to α our assertion is proved.

We now proceed with the proof of Theorem 1. We assume the existence of a transcendental meromorphic function f such that f'/f is transcendental with lower order $\mu < \alpha < 1/2$, and we assume that

$$\delta(\infty, f) > 1 - \sigma > \mu/\alpha, \quad \delta(0, f'/f) > 1 - \cos \pi\alpha. \tag{10}$$

Let H be the set of positive r such that the circle $|z| = r$ contains a multiple point of f, or meets the level curves $|f(z)| = 1$ infinitely often, or is at some point tangent to one of the level curves $|f(z)| = 1$. Then H consists only of isolated values and, provided r is not in H we have, simply differentiating $m(r, f) + N(r, f)$,

$$r \frac{d}{dr} (T(r, f)) = \frac{1}{2\pi} \int_{\theta : |f(re^{i\theta})| > 1} r \frac{\partial}{\partial r} \left(\log |f(re^{i\theta})| \right) d\theta + n(r, f) =$$

$$= \frac{1}{2\pi} \int_{\theta : |f(re^{i\theta})| > 1} Re \left(re^{i\theta} f'(re^{i\theta}) / f(re^{i\theta}) \right) d\theta + n(r, f). \tag{11}$$

But, by Cartan's formula [13, p. 8],

$$r \frac{d}{dr} (T(r, f)) = h(r) = \frac{1}{2\pi} \int_0^{2\pi} n(r, e^{i\phi}, f) d\phi. \tag{12}$$

This function $h(r)$ is non-decreasing. By (10) and (12) we have

$$\int_1^r n(t, f) dt/t \leq \int_1^r \sigma h(t) dt/t + O(1).$$

Choose $K > 1$ such that

$$\sigma < 1/K < 1 - \mu/\alpha \ . \tag{13}$$

This is possible, by (10). Now Lemma 3 of [1] (see also [2]) implies that there is a set E of lower logarithmic density $1 - 1/K$ such that, for all $r > 1$ with $r \in E$, we have

$$n(r, f) \leq K\sigma h(r). \tag{14}$$

We apply the remarks at the beginning of this section to $g = f/f'$. We obtain a set F as in (9) of upper logarithmic density at least $1 - \mu/\alpha$ such that, for large r with $r \in F$, we have, with c_α a positive constant,

$$\log |f(z)/f'(z)| > c_\alpha T(r, f/f') > 3 \log r \tag{15}$$

on $|z| = r$. Since $1/K < 1 - \mu/\alpha$, (11), (12), (14) and (15) give us arbitrarily large r such that

$$h(r) \leq O(r^{-2}) + K\sigma h(r).$$

But (13) implies that $K\sigma < 1$, and we have a contradiction, as $T(r, f) \leq h(r) \log r + O(1)$. This completes the proof of Theorem 1.

We sketch here how to prove an analogue of Theorem 1 for delta-subharmonic functions of the form (1). In an annulus $s_1 < |z| < s_2$ containing none of the z_k, the function u is harmonic, and is locally the real part of a function of form $H_1(z) + \lambda \log z$, with H_1 analytic and λ a constant. The characteristic function of u is (see, for example, p.354 of [18])

$$T(r, u) = \frac{1}{2\pi} \int_{\theta: u(re^{i\theta}) \geq 0} u(re^{i\theta}) d\theta + N(r, u_2) =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \max\{u_1(re^{i\theta}), u_2(re^{i\theta})\} d\theta,$$

in which we have written u as a difference of sums of the form (1), each with positive coefficients a_k. The characteristic is an increasing convex function of $\log r$, and we have, for r outside a set of isolated values,

$$r(d/dr)\,(T(r, u)) = \frac{1}{2\pi} \int_{\theta: u(re^{i\theta}) \geq 0} Re\left(re^{i\theta} G(re^{i\theta})\right) d\theta + n(r, u_2),$$

with G as in (2) and $n(r, u_2)$ just the sum of the absolute values of those a_k with $|z_k| \leq r$ and $a_k < 0$. Assuming that G has lower order $\mu < \alpha < 1/2$ and that $\delta(0, G) > 1 - \cos \pi\alpha$, while

$$\limsup_{r \to \infty} N(r, u_2)/T(r, u) < \sigma < 1 - \mu/\alpha,$$

then we choose K as in (13). We obtain $n(r, u_2) < K\sigma r dT/dr$ on a set of lower logarithmic density $1 - 1/K$, while G is small on a set of circles having upper logarithmic density at least $1 - \mu/\alpha$. Thus $dT/dr = O(r^{-2})$ through a sequence tending to infinity, contradicting the convexity of $T(r, u)$.

3 Lemmas needed for Theorems 2 and 3

For the following lemma we refer the reader to Theorem 8 of [14] and the discussion on p.141 thereof.

Lemma 1 *Suppose that G is transcendental and meromorphic in the plane such that*

$$m(r, G) - r^{1/2} \int_r^\infty \frac{n(t, G)}{t^{3/2}} dt \to +\infty \tag{16}$$

as $r \to \infty$*, and suppose that* $|G(z)| \leq M < \infty$ *on a path* Γ *tending to infinity. Then there exists a function* v*, non-constant, non-negative and subharmonic in the plane, such that* $v(z) \leq \log^+ |G(z)| + O(1)$ *for all* z*.*

We remark that Hayman proved in [14] that (16) suffices to ensure that ∞ is an asymptotic value of G.

Lemma 2 *Suppose that* $d_1 > 0$ *and* $d \geq 1$ *and that* G *is transcendental and meromorphic in the plane, such that* $T(r, G) \leq d_1 r^d$ *for all large* r*. Then there exist arbitrarily large positive* R *with the following properties.* $G(z)$ *has no critical values on* $|w| = R$ *and, if* $L(r, R, G)$ *denotes the total length of the level curves* $|G(z)| = R$ *lying in* $|z| < r$*, then we have*

$$L(r, R, G) \leq c r^{(3+d)/2}, \quad r \geq (1/4)\log R, \tag{17}$$

in which c *is a positive constant depending only on* d *and* d_1*. Further, if* $r^* \geq 1$*, and* S^* *denotes the union of the level curves* $|G(z)| = R$ *lying in* $|z| \geq r^*$*, we have*

$$\int_{S^*} |z^{-(5+d)/2} G(z)^{-1}| \, |dz| \leq C \left((r^*)^{-(5+d)/2} (\log R)^{(3+d)/2} R^{-1} + (r^*)^{-1} R^{-1} \right), \tag{18}$$

in which C *is a positive constant again depending only on* d *and* d_1*.*

Proof Using the length-area inequality as in [21] we have, for $r > 0$,

$$\int_0^\infty \frac{L(r, R, G)^2}{p(r, R, G) R} dR \leq 2\pi^2 r^2, \tag{19}$$

in which

$$p(r, R, G) = \frac{1}{2\pi} \int_0^{2\pi} n(r, Re^{i\phi}, G) d\phi.$$

Assume that S is large, in particular so large that $T(4r, G) \leq d_1 (4r)^d$ for $r \geq (1/4)\log S$. We have, by (19),

$$\int_{(1/4)\log S}^\infty \int_S^{2S} \frac{L(r, R, G)^2}{p(r, R, G)} \, \frac{dR}{R} \, \frac{dr}{r^4} < 1. \tag{20}$$

But for ϕ real and for $S \leq R \leq 2S$, and $r \geq (1/4)\log S$, we have $\infty \geq |G(0) - Re^{i\phi}| \geq 1$ and

$$n(r, Re^{i\phi}, G) \leq N(4r, 1/(G - Re^{i\phi})) \leq T(4r, G - Re^{i\phi}) + C_1 \leq$$

$$\leq T(4r, G) + \log R + \log 2 + C_1 < 2^{2d+1} d_1 r^d$$

provided S is large enough, the constant C_1 only arising if $G(0) = \infty$. Now (20) gives, on changing the order of integration,

$$\int_S^{2S} \int_{(1/4)\log S}^{\infty} L(r, R, G)^2 \, \frac{dr}{2^{2d+1} d_1 r^{4+d}} \, \frac{dR}{R} < 1.$$

Thus there exists R with $S < R < 2S$ such that

$$\int_{(1/4)\log S}^{\infty} L(r, R, G)^2 \, \frac{dr}{2^{2d+1} d_1 r^{4+d}} < (\log 2)^{-1}.$$

This R may be chosen so that the level curves $|G(z)| = R$ have no multiple points. Since, for fixed R, the quantity $L(r, R, G)$ increases with r, we obtain (17).

Now suppose that R satisfies (17), and that $r^* \geq 1$, and S^* denotes the union of the level curves $|G(z)| = R$ lying in $|z| \geq r^*$. If $r^* < T = (1/4)\log R$, then

$$\int_{S^* \cap \{r* \leq |z| < T\}} |z^{-(5+d)/2} G(z)^{-1}| \, |dz| \leq c T^{(3+d)/2} (r^*)^{-(5+d)/2} R^{-1}.$$

Suppose now that $r^* \geq T$. Then

$$\int_{S^*} |z^{-(5+d)/2} G(z)^{-1}| \, |dz| \leq$$

$$\leq c \sum_{n=0}^{\infty} (2^{n+1} r^*)^{(3+d)/2} (2^n r^*)^{-(5+d)/2} R^{-1} = 2^{(5+d)/2} c (r^*)^{-1} R^{-1}.$$

Thus, whether or not $r^* \geq T$, we obtain (18).

For convenience later on, we state the following as a lemma. It is a simple consequence of the Jordan curve theorem.

Lemma 3 *Let G be a domain, whose boundary consists of countably many disjoint analytic curves, each either simple closed or simple and going to infinity in both directions, let z_1, z_2 be in G, and let η be a simple path from z_1 to z_2 which meets ∂G only finitely often and is never tangent to any of the boundary curves of G. Then there exists a simple path σ from z_1 to z_2 such that σ is contained in $G \cup \partial G$ and consists of sub-paths of η and arcs of boundary curves of G.*

The next lemma uses several of the ideas from Lemma 4 of [14].

Lemma 4 *Suppose that G is transcendental and meromorphic in the plane such that $|G(z)| \to \infty$ on a path γ tending to infinity, and suppose that $G(z)$ has no* *critical values on $|w| = R > 0$. Suppose further that $s_1 > 1$, and that S^* is the union of those level curves $|G(z)| = R$ lying in $|z| \geq s_1$. Here we assume that if a level curve $|G(z)| = R$ meets both $|z| \geq s_1$ and $|z| < s_1$ then S^* contains only the part lying in $|z| \geq s_1$. Then at least one of the following is true.*

(a) There is a simple path $\Gamma_0(G)$ tending to infinity on which $|G(z)| = R$.

(b) There exist a simple path $\Gamma_1(G)$ tending to infinity and simple closed curves τ_n surrounding the origin such that:

(i) $\Gamma_1(G)$ is contained in the union of S^* and a ray $z = re^{i\psi}, r \geq s_1$, ψ real;

(ii) For each n there exists $r_n > 0$ such that τ_n is contained in the union of S^* and the circle $|z| = r_n$, and $|z| \geq r_n$ for all z on τ_n, while $r_n \to \infty$ as $n \to \infty$;

(iii) we have $|G(z)| \geq R$ on $\Gamma_1(G)$ and on the τ_n.

Proof We first note that since G is transcendental any circle $|z| = r > 0$ contains at most finitely many points at which $|G(z)| = R$. Now, if there exists an unbounded level curve $|G(z)| = R$, then we obviously have conclusion (a), and we assume henceforth that all level curves $|G(z)| = R$ are bounded, so that each is a simple closed analytic curve, and we denote these curves by γ_μ. We choose $s_2 > s_1$ so large that $|G(z)| > R$ on the part of γ lying in $|z| > s_2$. Now, only finitely many γ_μ can meet $|z| \leq s_2$. Further, any γ_μ which does not meet $|z| \leq s_2$ cannot surround the origin, because it cannot meet γ. We choose $s_3 > s_2$ and denote by S^{**} the union of those γ_μ which meet $|z| \geq s_3$, and we assume that s_3 is so large that S^{**} does not meet $|z| \leq s_2$. Then S^{**} is clearly contained in S^* and consists of simple closed analytic curves which close in $|z| > s_2$ and do not surround the origin.

We take a sequence $r_n \to \infty$ and we take a real ψ such that no level curve $|G(z)| = R$ is tangent to either the ray $\arg z = \psi$ or to any of the circles $|z| = r_n$. This may be ensured by parametrizing the level curves $|G(z)| = R$ with respect to $\arg G(z)$ and considering the stationary points of $\log |z|$ and $\arg z$ on these analytic curves. We can also assume that S^* has no finite limit points on the ray $\arg z = \psi$, because otherwise we would have $|G(z)| = R$ on the whole ray, and conclusion (a). Now the set $\{z : |G(z)| > R\}$ has an unbounded component, meeting γ, and so all components U_j of the set $\{z : |G(z)| < R\}$ must be bounded. We denote the unique unbounded component of $\{z : |G(z)| > R\}$ by V.

We show that conclusion (b) holds, as follows. We choose a sequence of points $z_n \in V$ such that $\arg z_n = \psi$ and $|z_n|$ is increasing to ∞. We apply Lemma 3 to the domain V, obtaining a path from z_n to z_{n+1}, and the union of these forms a path tending to infinity, which we can replace, if necessary, by a simple path.

We now form the curves τ_n surrounding the origin, as follows. For n large, we take the domain V_n consisting of the union of $\{z : |z| < r_n\}$ and all U_j which meet $|z| = r_n$, and τ_n is simply the external boundary of V_n.

The next lemma is standard, $B(r,u)$ being defined by $B(r,u) = \sup\{u(z) : |z| = r\}$.

Lemma 5 *Let v_1, v_2 be non-constant subharmonic functions in the plane, let s_1, s_2 be real constants, and let U_1, U_2 be disjoint, unbounded domains such that $v_j \leq s_j$ on ∂U_j and $v_j(z_j) > s_j$ for at least one point z_j in U_j. Then*

$$\log B(r, v_1) + \log B(r, v_2) \geq 2 \log r - O(1)$$

for all large r.

Proof We set $u_j(z) = \max\{v_j(z) - s_j, 0\}$ for $z \in U_j$, and $u_j(z) = 0$ otherwise. For r large, let $\theta_j(r)$ be the angular measure of the intersection of U_j with $|z| = r$. Then for some r_0 and for all sufficiently large r we have (see p.116 of [20])

$$\pi \int_{r_0}^{r/2} \frac{dt}{t\theta_j(t)} \leq \log B(r, u_j) + O(1).$$

But we have $\theta_1(t)^{-1} + \theta_2(t)^{-1} \geq 4/(\theta_1(t) + \theta_2(t)) \geq 2/\pi$, and this proves the lemma.

4 Proof of Theorem 3

Suppose that d is an odd positive integer and that f is transcendental and meromorphic in the plane such that $T(r,f) = O(r^d)$ as $r \to \infty$, while

$$\liminf_{r \to \infty} T(r,f)/r = 0.$$

We set $N = (5+d)/2$ and $h(z) = 1/z^N f'(z)$, and assume that

$$m(r,h) - r^{1/2} \int_r^\infty \frac{n(t,h)}{t^{3/2}} dt \to \infty \tag{21}$$

as $r \to \infty$. We choose a large positive R satisfying the conclusions of Lemma 2, with $G = h$. By (21) and [14], there exists a path tending to infinity on which $|h(z)| \to \infty$ and so, applying Lemma 4, there are two possibilities. Either conclusion (a) holds, in which case we obtain a path $\Gamma_0(h)$ tending to infinity on which $|h(z)| = R$, and we can assume that $\Gamma_0(h)$ lies in $|z| \geq 1$: by Lemma 2 and the definition of h,

$$\int_{\Gamma_0(h)} |f'(t)| \, |dt| < \infty,$$

so that there is a finite value α such that $f(z) \to \alpha$ as $z \to \infty$ on $\Gamma_0(h)$.

The alternative possibility is that we have conclusion (b) of Lemma 4, in which case we obtain a path $\Gamma_1(h)$ as in that lemma, lying in $|z| \geq 1$ and tending to infinity. Now, if γ_1 is the part of $\Gamma_1(h)$ contained in the union of the level curves $|h(z)| = R$ lying in $|z| \geq 1$, then we have, by Lemma 2,

$$\int_{\gamma_1} |f'(t)|\, |dt| < \infty.$$

Further, if γ_2 is the part of $\Gamma_1(h)$ contained in the ray $z = re^{i\psi}$, $r \geq 1$, then we have

$$\int_{\gamma_2} |f'(t)|\, |dt| \leq \int_1^\infty R^{-1} t^{-N}\, dt < \infty.$$

Again, we conclude that $f(z)$ tends to a finite value, α say, as $z \to \infty$ on $\Gamma_1(h)$.

Moreover, if conclusion (b) holds, we consider

$$I_n = \int_{\tau_n} |f'(t)|\, |dt|,$$

with τ_n the closed curves as in Lemma 4. Now the contribution to I_n from the part of τ_n which is contained in $|z| = r_n$ is at most $2\pi r_n^{-N+1} R^{-1}$, while the contribution to I_n from the rest of τ_n is, using Lemma 2, at most

$$O\left(r_n^{-N} (\log R)^{(3+d)/2} R^{-1} + r_n^{-1} R^{-1}\right).$$

Thus $f(z) = \alpha + o(1)$ as $z \to \infty$ in the union of $\Gamma_1(h)$ and the τ_n.

We now set $H(z) = (f(z) - \alpha)/z^N f'(z)$, and we assume that

$$m(r, H) - r^{1/2} \int_r^\infty \frac{n(t, H)}{t^{3/2}} dt \to \infty \tag{22}$$

as $r \to \infty$. We can choose R' large and positive, such that $H(z)$ has no critical values on $|w| = R'$, and such that the conclusion of Lemma 2 holds, with $G = H$. We apply Lemma 4, and in either case (a) or (b) we obtain a path $\Gamma_2(H)$ tending to infinity such that

$$\int_{\Gamma_2(H)} |f'(t)/(f(t) - \alpha)|\, |dt| < \infty.$$

It follows that there is some non-zero constant α' such that $f(z) - \alpha \to \alpha'$ as $z \to \infty$ on $\Gamma_2(H)$. In order that this be compatible with our conclusions for h, we must have conclusion (a) of Lemma 4 for h.

Thus we have a path $\Gamma_0(h)$ tending to infinity on which $|h(z)| = R$ and $f(z) \to \alpha$. Applying Lemma 1, we see from (21) that there exists a non-constant non-negative subharmonic function v_h such that

$$v_h(z) \leq \log^+ |h(z)| + c_1 \tag{23}$$

for all z. Further [17], there is a path $\gamma_0(h)$ lying in $|z| \geq 1$ and tending to infinity such that $v_h(z) \to \infty$ as $z \to \infty$ on $\gamma_0(h)$ and such that

$$\int_{\gamma_0(h)} \exp(\ -v_h(z) \ ) \ |dz| < \infty,$$

so that, by (23), we can assume that

$$\int_{\gamma_0(h)} |f'(z)| \ |dz| < \infty.$$

Hence there is some constant β such that $f(z) \to \beta$ as $z \to \infty$ on $\gamma_0(h)$. We set $g(z) = (f(z) - \beta)/z^N f'(z)$, and assume that

$$m(r,g) - r^{1/2} \int_r^\infty \frac{n(t,g)}{t^{3/2}} dt \to \infty \tag{24}$$

as $r \to \infty$. On the path $\Gamma_0(h)$ we have $|h(z)| = R$ and $f(z) \to \alpha$, and so $|g(z)| \leq c_2$ on $\Gamma_0(h)$. Applying Lemma 1 to g we now obtain, in view of (24), a function $v_g(z)$ non-constant, non-negative and subharmonic in the plane such that

$$v_g(z) \leq \log^+ |g(z)| + c_3$$

for all z. We obtain, in addition, a path $\gamma_0(g)$, lying in $|z| \geq 1$ and tending to infinity, on which $v_g(z) \to \infty$, and such that

$$\int_{\gamma_0(g)} \exp(\ -v_g(z) \ ) \ |dz| < \infty$$

and hence

$$\int_{\gamma_0(g)} |f'(z)/(f(z) - \beta)| \ |dz| < \infty,$$

which implies that $f(z) - \beta$ tends to some finite, non-zero value β' as $z \to \infty$ on $\gamma_0(g)$.

We choose s^* large, with $f'(z) \neq 0$ on $|z| = s^*$, and we choose S and S' large, with

$$S > M(s^*, h), \quad S' > M(s^*, g), \tag{25}$$

such that S satisfies the conclusions of Lemma 2, with $G = h$, as does S', with $G = g$. Now for z on $\gamma_0(h)$ with $|z|$ large we have $v_h(z) > 2 \log S$, and so (23) implies that this part of $\gamma_0(h)$ lies inside an unbounded component $U_1(h)$ of the set $\{z : |h(z)| > S\}$. On the boundary of $U_1(h)$ we have $v_h(z) < 2 \log S$, and $U_1(h)$ lies in $|z| > s^*$, because of (25). Similarly, an unbounded sub-path of $\gamma_0(g)$ lies inside an unbounded component $U_1(g)$ of the set $\{z : |g(z)| > S'\}$, and again $U_1(g)$ lies in $|z| > s^*$, with $v_g(z) < 2 \log S'$ on the boundary of $U_1(h)$.

We estimate $|f(z_1) - \beta|$, for $z_1 \in U_1(h)$. We choose $z_2 \in U_1(h)$, lying on $\gamma_0(h)$, such that $|z_2|$ is large and $|f(z_2) - \beta| < |\beta'|/8$. We join z_1 to z_2 by a simple path η which consists of part of the circle $|z| = |z_2|$ and part of the ray $\arg z = \arg z_1, |z| \geq |z_1|$. We assume that η meets the boundary of $U_1(h)$ only finitely often, and is never tangent to any level curve $|h(z)| = S$, displacing z_1 and z_2 slightly, if necessary, to achieve this. We then use Lemma 3 to form a curve σ from z_1 to z_2 consisting of part of η and part of the boundary of $U_1(h)$, such that on this path we have $|h(z)| \geq S$. The contribution to

$$I^* = \int_\sigma |f'(z)||dz|$$

from the circle $|z| = |z_2|$ is at most $2\pi S^{-1}|z_2|^{-N+1}$, while the contribution from the ray $\arg z = \arg z_1$ is at most

$$\int_{|z_1|}^{\infty} S^{-1}t^{-N}dt \leq S^{-1}(s^*)^{-N+1}.$$

Finally, the contribution to I^* from the boundary of $U_1(h)$ is, by Lemma 2, at most

$$O\left((s^*)^{-N}(\log S)^{(3+d)/2}S^{-1} + (s^*)^{-1}S^{-1}\right).$$

It follows that, provided s^* was chosen large enough, we have $|f(z)-\beta| < |\beta'|/4$ on $U_1(h)$. Similar reasoning gives us $|f(z) - \beta - \beta'| < |\beta'|/4$ on $U_1(g)$, and we conclude that $U_1(h)$ and $U_1(g)$ are disjoint.

However, by (23), the growth of v_h may be estimated by writing

$$B(r, v_h) \leq \frac{3}{2\pi}\int_0^{2\pi} v_h(2re^{i\phi})d\phi \leq 3m(2r, h) + O(1) \leq$$

$$\leq O(T(2r, f) + \log r).$$

The growth of v_g may be estimated similarly, and we have a contradiction, applying Lemma 5 to $v_h, v_g, U_1(h), U_1(g)$.

It follows that one of the assumptions (21), (22), (24) must be false, and this gives us (6) or (7), using the fact that $m(r, 1/f') \leq m(r, h) + N\log r$ and $n(t, h) \leq n(t, 1/f') + N$.

5 Proof of Theorem 2

We assume the existence of a transcendental meromorphic function f such that $T(r, f) = O(r^d)$ as $r \to \infty$, for some odd positive integer d, and such that $\liminf_{r\to\infty} T(r, f)/r = 0$, and we assume that both (4) and (5) are false. We set $h(z) = f(z)/z^N f'(z)$, again with $N = (5 + d)/2$. Then by [14], there are paths γ_h, γ_f tending to infinity, such that $h(z) \to \infty$ as $z \to \infty$ on γ_h and $f(z) \to \infty$ on γ_f.

We choose a large positive R such that the conclusions of Lemma 2 hold, with $G = h$, and we apply Lemma 4. This gives us either a path $\Gamma_0(h)$ tending to infinity on which $|h(z)| = R$ or, if conclusion (b) holds, a path $\Gamma_1(h)$ tending to infinity on which $|h(z)| \geq R$ and a family of closed curves τ_n surrounding the origin. In the first case, Lemma 2 implies that f is bounded on $\Gamma_0(h)$, while if conclusion (b) holds we see that f is bounded on the union of $\Gamma_1(h)$ and the τ_n. The latter possibility contradicts the existence of the path γ_h, and so we must have conclusion (a), and both h and f are bounded on the path $\Gamma_0(h)$.

We now apply Lemma 1 to h, to obtain a function v_h non-constant, non-negative and subharmonic in the plane, such that $v_h(z) \leq \log^+ |h(z)| + c_1$, and a path $\gamma_0(h)$, lying in $|z| \geq 1$ and tending to infinity, such that $v_h(z) \to +\infty$ as $z \to \infty$ on $\gamma_0(h)$, and

$$\int_{\gamma_0(h)} \exp(\; - v_h(z) \;) \; |dz| < \infty,$$

so that we may assume that

$$\int_{\gamma_0(h)} |f'(z)/f(z)| \; |dz| < \infty.$$

Thus there is some constant β such that $f(z) \to \beta$ as $z \to \infty$ on $\gamma_0(h)$. We again choose s^* large and $S > M(s^*, h)$ such that S satisfies the conclusions of Lemma 2, with $G = h$, and the path $\gamma_0(h)$, on which $v_h(z) \to +\infty$, must have unbounded intersection with an unbounded component $U_1(h)$ of the set $\{z : |h(z)| > S\}$. Again, $v_h(z) < 2\log S$ on the boundary of $U_1(h)$, and $U_1(h)$ lies in $|z| > s^*$. Reasoning as in the proof of Theorem 3 and applying Lemma 2 again we see that $|f(z)| \leq c_2$, say, on $U_1(h)$.

But we can apply Lemma 1 to f to obtain a non-constant subharmonic function v_f such that $v_f(z) \leq \log^+ |f(z)| + c_3$ for all z. We choose S' large and a path tending to infinity on which $v_f(z) > 2\log S'$, which must lie inside an unbounded component $U_1(f)$ of the set $\{z : |f(z)| > S'\}$. On the boundary of $U_1(f)$ we have $|f(z)| = S'$ and $v_f(z) < 2\log S'$, and $U_1(f)$ cannot meet $U_1(h)$, if S' is large enough. Applying Lemma 5 to v_h and v_f we have a contradiction.

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Department of Mathematics, University of Nottingham, NG7 2RD.

Department of Mathematics, University of Wisconsin, 53706.