Lemma 0.1 *Let G be a domain in $\mathbb{C}$, whose boundary ∂G consists of countably many pairwise disjoint curves γ_k, each either simple closed or simple and going to infinity in both directions, and assume that these do not accumulate at finite points in the following sense: each $a \in \mathbb{C}$ has $r_a > 0$ such that the open disc $B(a, r_a)$ meets at most one γ_k. Let z_1, z_2 be in G, and let η be a simple path from z_1 to z_2. Then there exists a simple path σ from z_1 to z_2 such that σ is contained in $G \cup \partial G$ and consists of sub-paths of η and arcs of boundary curves of G.*

Note that the hypotheses of the lemma are clearly satisfied if G is a component of the set $\{z \in \mathbb{C} : |f(z)| < R\}$, where f is a meromorphic function on $\mathbb{C}$ and R is such that $|f(z)| = R$ implies $f'(z) \neq 0$.

Proof. Assume that η is defined on $I = [0, 1]$ and that η meets ∂G, since otherwise there is nothing to prove. By compactness, η is covered by finitely many open discs $B(a, r_a/2)$ such that each $B(a, r_a)$ meets at most one γ_k. Hence there exists $\varepsilon > 0$ (the minimum of finitely many $r_a/2$) such that if $a \in \eta$ then $B(a, \varepsilon)$ meets at most one γ_k. Moreover, there exists $\delta > 0$ such that $|\eta(s) - \eta(t)| < \varepsilon$ for all $s, t \in I$ with $|s - t| \leq \delta$. If γ_k is bounded, then by the Jordan curve theorem its complement in $\mathbb{C}$ consists of two disjoint domains, E_k and F_k. On the other hand, if γ_k is unbounded, adjoin to it the point at infinity, so that this time the complement of γ_k on the Riemann sphere consists of two disjoint domains, E_k and F_k. For each k, whether or not γ_k is bounded, we may assume that $G \subseteq E_k$.

Thus $G \subseteq E = \cap E_k$, and E is a connected subset of $\mathbb{C}$, by the chaining lemma. Moreover, E is open: to see this, take $a \in E$. Then $a \in \mathbb{C}$ and there exists $r > 0$ such that $B(a, r)$ meets at most one γ_k. Hence $B(a, r)$ lies in all but at most one E_k, and reducing r if necessary gives an open disc of centre a lying in E. It follows that E is a domain, with $G \subseteq E$. Suppose that $G \neq E$. Then there exists a path in E joining $z_1 \in G$ to $z_3 \notin G$, and this path must meet ∂G and so meet some γ_k, a contradiction. Hence $G = E$.

Because $z_1 \in G$ and ∂G is closed, $t_0 = \min\{t \in [0, 1] : \eta(t) \in \partial G\}$ exists and is positive: choose γ_k such that $\eta(t_0) \in \gamma_k$, and let $t_1 = \max\{t \in [0, 1] : \eta(t) \in \gamma_k\}$. Since $z_1, z_2 \in G \subseteq E_k$, we have $\eta(t) \in E_k$ for $0 \leq t < t_0$ and $t_1 < t \leq 1$. Choose a bounded arc λ of γ_k joining $\eta(t_0)$ to $\eta(t_1)$ and replace the part of η for $t_0 \leq t \leq t_1$ by λ. By the choice of t_0, t_1, this does not affect the injectivity of the path.

For $t_1 < t \leq t_2 = \min\{t_1 + \delta, 1\}$ we have $\eta(t) \in E_k \cap B(\eta(t_1), \varepsilon)$ and so $\eta(t) \notin \bigcup \gamma_j$, by the choice of ε and t_1. Assume that $t_3 \in (t_1, t_2]$ has $\eta(t_3) \notin G$. Then $\eta(t_3) \notin E$, and so there exists $m \neq k$ with $\eta(t_3) \notin E_m$ and hence $\eta(t_3) \in F_m$. Since $\eta(t) \notin \gamma_m$ for $t_1 \leq t \leq t_3$, we must have $\eta(t_1) \in F_m$. But $\eta(t_1) \in \gamma_k \subseteq \partial G$, and so G meets F_m, a contradiction.

Hence we have $\eta(t) \in G$ for $t_1 < t \leq t_2$. If $\eta(t) \notin G$ for some $t \in [t_2, 1]$ then $t_2 < 1$; hence we may take $t_4 = \min\{t \in [t_2, 1] : \eta(t) \in \partial G\}$ and repeat the process, but the fact that $t_4 - t_1 \geq \delta$ means that repetition cannot occur infinitely many times.